Econometrica Supplementary Material

SUPPLEMENT TO "A THEORY OF DISAGREEMENT IN REPEATED GAMES WITH BARGAINING"

(*Econometrica*, Vol. 81, No. 6, November 2013, 2303–2350)

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APPENDIX B

B.1. *Efficiency and Bargaining Power*

THE FOLLOWING THREE RESULTS PROVIDE sufficient conditions and explicit guidance for constructing efficient contractual equilibria when the best response correspondences satisfy certain conditions.

THEOREM 9: *For each i, let a^i be a pure action profile in the stage game. If a^i_i is a best response to a^i_{-i} for both i and $(\pi_2, -\pi_1)\cdot(u(a^2)-u(a^1))>0$, then the following two-state automaton strategy, with states 1 and 2, yields a BSG set for δ sufficiently high*:

- *Disagreement*: *In state i, make no transfers and play a^i in the action phase. If player $j \neq i$ deviates unilaterally, go to state j; otherwise stay in state i.*
- *Agreement*: *Play $\arg\max_a \sum_i u_i(a)$ in the action phase and pay transfers that split the surplus with respect to disagreement play in the current state. If nobody deviates or both deviate, randomize between the two states with equal probabilities. If player i deviates unilaterally, go to state i.*

The continuation values in the two states are the endpoints of a BSG set contained in $V(S_{\mathrm{CE}})$.

This works because player i needs no incentives at a^i, while player $-i$ can be given strong incentives by the threat of switching to state $-i$. So under disagreement, if nobody deviates, they can stay in the same state. This being the case, their agreement utility in each state is on a π_2/π_1-sloped line from their stage-game payoff under disagreement.

Since a player who is being minimaxed is always playing a best response in the stage game, this implies a minimax separation condition that may be easy to check in many games.

COROLLARY 3: *Let ζ^i be the pure action minimax payoff profile for player i in the stage game. Suppose that $(\pi_2, -\pi_1)\cdot(\zeta^2-\zeta^1)>0$. Then contractual equilibrium attains efficiency if the players are sufficiently patient.*

Our next result demonstrates that patient players can attain efficiency in a contractual equilibrium if the stage game is continuous with an interior Nash equilibrium around which the best response functions are differentiable, and an increase in one player's action strictly reduces both the other player's best response and his stage-game payoff.

 DOI: 10.3982/ECTA10361

THEOREM 10: *Suppose that* $u:\mathcal{A}\to\mathbb{R}^2$ *is uniformly bounded, and, for each player* i, *there exists an open interval* $(\underline{a}_i,\overline{a}_i)\subset\mathcal{A}_i$ *such that* (i) u *is twice differentiable on* $(\underline{a}_1,\overline{a}_1)\times(\underline{a}_2,\overline{a}_2)$, (ii) *each player* i*'s best response function* BR_i *is differentiable on* $(\underline{a}_1,\overline{a}_1)\times(\underline{a}_2,\overline{a}_2)$, *and* (iii) *there exists a stage-game Nash equilibrium* $a^{\mathrm{NE}}\in(\underline{a}_1,\overline{a}_1)\times(\underline{a}_2,\overline{a}_2)$. *If, for both* i, $dBR_i(a_{-i})/da_{-i}<1$ *and* $du_i(a)/da_{-i}<0$ *at* a^{NE}, *then contractual equilibrium attains efficiency if the players are sufficiently patient.*

As an example, the symmetric Cournot duopoly game does not satisfy the conditions of Corollary 3, since both firms earn zero profits whenever one firm is minimaxed. However, it has an interior Nash equilibrium and a uniformly bounded profit function, its best response functions have negative slope, and each firm's payoff is decreasing in the other's quantity. That is, it satisfies the conditions of Theorem 10 and, therefore, has an efficient (collusive) contractual equilibrium if the firms are sufficiently patient.

The remainder of this section proves these results, along with Theorem 6 from the main text.

LEMMA 12: *Given any* $\delta,\delta'\in(0,1)$ *and* $d\geq 0$, *let* $d'\equiv\frac{\delta(1-\delta')}{\delta'(1-\delta)}d$. *Then* $\Gamma_{\delta'}(d')=\Gamma_\delta(d)$.

PROOF: Consider the definition of γ^i_δ (see Eq. (6)), and suppose that α and η satisfy the constraints for discount factor δ. Let $\eta'\equiv\frac{\delta(1-\delta')}{\delta'(1-\delta)}\eta$. Note that the first constraint is equivalent to $\eta'\in[-d',0]$. Regarding the second constraint, observe that

$$(1-\delta)u+\delta(\eta,-\eta)=\frac{1-\delta}{1-\delta'}\big((1-\delta')u+\delta'(\eta',-\eta')\big). \tag{24}$$

Thus, α and η satisfy the second constraint for δ if and only if α and η' satisfy the same constraint for δ'. Finally, note that the value of the objective function of Eq. (6) at α, η, and δ is equal to the value at α, η' and δ'. These facts imply that $\gamma^i_\delta(d)=\gamma^i_{\delta'}(d')$ and so $\Gamma^\delta(d)=\Gamma^{\delta'}(d')$. *Q.E.D.*

PROOF OF THEOREM 6: Fix $\hat\delta$. If $\Gamma_{\hat\delta}(\infty)>0$, then there exists a number $\hat d>0$ such that $\Gamma_{\hat\delta}(\hat d)>0$. For any δ', let $d'\equiv\frac{\hat\delta(1-\delta')}{\delta'(1-\hat\delta)}\hat d$. From Lemma 12, we see that $\Gamma_{\delta'}(d')=\Gamma_{\hat\delta}(\hat d)$. As δ' converges to 1, d' converges to 0, which implies that the maximal fixed point of Γ_δ is bounded below by $\Gamma_{\hat\delta}(\hat d)$ for sufficiently high δ. Finally, it is clear from the definition of λ^* that if d^* is bounded away from 0 for large discount factors, then any stage-game action profile can be supported for δ sufficiently large. This proves the first statement of the theorem. Regarding the second statement, observe that $\Gamma_\delta(\infty)=0$ implies that

$\Gamma_\delta(d) = 0$ for all δ and d. Thus, the maximal fixed point of Γ is 0 regardless of δ and only stage-game Nash equilibrium action profiles can be supported. *Q.E.D.*

PROOF OF THEOREM 9: Let $a^* \in \arg\max_a \sum_i u_i(a)$. Under these strategies, the continuation value in state i is $\hat{z}^i \equiv u(a^i) + \pi \sum_j (u_j(a^*) - u_j(a^i))$. Note that $\hat{z}^i$ does not depend on δ. Since $(\pi_2, -\pi_1) \cdot (u(a^2) - u(a^1)) > 0$, the payoff span $\hat{z}_1^2 - \hat{z}_1^1$ is strictly positive and constant in δ. We must check that the sequential rationality constraints are satisfied. Under disagreement in state i, player i is playing a stage-game best response and anticipates remaining in state i regardless of her action. Under agreement in either state, player i anticipates a loss of $\frac{\delta}{1-\delta}\frac{1}{2}(\hat{z}_1^2 - \hat{z}_1^1)$ if she deviates. Similarly, under disagreement in state $-i$, player i anticipates a loss of $\frac{\delta}{1-\delta}(\hat{z}_1^2 - \hat{z}_1^1)$ if she deviates. Hence efficiency is attained and all actions are sequentially rational for δ sufficiently high. Note that $\hat{z}^1$ and $\hat{z}^2$ are not necessarily the endpoints of $V(S_{\mathrm{CE}})$. *Q.E.D.*

PROOF OF THEOREM 10: It suffices to restrict attention to the stage game, and find a^1 and a^2 as described in Theorem 9. Choose a^{NE} satisfying the suppositions. For any small $\varepsilon' > 0$, let $a_i^{-i}(\varepsilon') \equiv a_i^{\mathrm{NE}} + \varepsilon'$ and $a_i^i(\varepsilon') \equiv BR_i(a_{-i}^i(\varepsilon'))$. Near a^{NE}, since u is twice differentiable, a^{NE} is an equilibrium, and BR_i is locally a differentiable function, for a_{-i} sufficiently close to a_{-i}^{NE}, it follows that $|u_i(a_i^i(\varepsilon'), a_{-i}) - u_i(a_i^{-i}(\varepsilon'), a_{-i})|$ is on the order of at most $O(\varepsilon'^2)$. Since $dBR_i/da_{-i} < 1$, it follows that $a_i^{-i} - a_i^i > 0$ is on the order of at least $O(\varepsilon')$. Since $du_i/da_{-i} < 0$, for a_i sufficiently close to a_i^{NE}, it also follows that $u_i(a_i, a_{-i}^{-i}(\varepsilon')) - u_i(a_i, a_{-i}^i(\varepsilon')) > 0$ is on the order of at least $O(\varepsilon')$. Hence, for $\varepsilon' > 0$ sufficiently small, each player i strictly prefers a^{-i} to a^i. Since player i is best responding at a^i, the conditions of Theorem 9 are satisfied. *Q.E.D.*

PROOF OF THEOREM 7: Define

$$
\begin{aligned}
(25)\qquad & \hat{\Gamma}(d, \alpha^1, \alpha^2) \\
&\equiv \max_{\eta^1, \eta^2} \Bigl(\pi_2\bigl(u_1(\alpha^1) - u_1(\alpha^2)\bigr) - \pi_1\bigl(u_2(\alpha^1) - u_2(\alpha^2)\bigr) \\
&\qquad + \frac{\delta}{1-\delta}\bigl(\eta^1(\alpha^1) - \eta^2(\alpha^2)\bigr)\Bigr) \\
&\text{s.t.} \quad \begin{cases} \eta^1 : \mathcal{A} \to [-d, 0], \text{ extended to } \hat{\Delta}\mathcal{A}, \\ \eta^2 : \mathcal{A} \to [0, d], \text{ extended to } \hat{\Delta}\mathcal{A}, \\ \alpha^1 \in \hat{\Delta}\mathcal{A} \text{ is a Nash equilibrium} \\ \quad \text{of } \bigl\langle \mathcal{A}, (1-\delta)u + \delta\bigl(\eta^1, -\eta^1\bigr)\bigr\rangle, \\ \alpha^2 \in \hat{\Delta}\mathcal{A} \text{ is a Nash equilibrium} \\ \quad \text{of } \bigl\langle \mathcal{A}, (1-\delta)u + \delta\bigl(\eta^2, -\eta^2\bigr)\bigr\rangle. \end{cases}
\end{aligned}
$$

That is, $\Gamma(d) = \max_{\alpha^1,\alpha^2} \hat{\Gamma}(d, \alpha^1, \alpha^2)$. Observe that the argmax (η^1, η^2) of $\hat{\Gamma}(d, \alpha^1, \alpha^2)$ is independent of π_1 and π_2. Hence $\hat{\Gamma}(d, \alpha^1, \alpha^2)$ is maximized at either $\pi = (0, 1)$ or $\pi = (1, 0)$, as is $\Gamma(d)$, as is $\max_d\{d : d = \Gamma(d)\}$. *Q.E.D.*

B.2. *Generalization*

The following theorem gives an explicit characterization of $\tilde{V}(S_{\mathrm{CE}})$ for the special case of two players, along the lines of Section 5.1, allowing for imperfect public monitoring and asymmetric discount factors $\boldsymbol{\delta} = (\delta_1, \delta_2)$.

THEOREM 11: *For a two-player game in the simplified form $\langle 2, \mathcal{A}, \Theta, f, u, \boldsymbol{\delta}, \pi\rangle$, if $\mathcal{A}$ and Θ are finite and $\tilde{W}^* = \tilde{V}(S_{\mathrm{CE}})$, then $\tilde{W}^*$ is a compact line segment of slope -1 that has the following properties*:

(i) *The* $\operatorname{span}(\tilde{W}^*)$ *is equal to the maximal fixed point of $\tilde{\Gamma} \equiv \tilde{\gamma}^2 + \tilde{\gamma}^1$, where, for players $i \neq j$,*

$$\psi \equiv \pi_1\delta_2 + \pi_2\delta_1, \tag{26}$$

$$\hat{u}(\alpha) \equiv \int_{\theta\in\Theta} u(\alpha, \theta)\, df(\theta|\alpha), \tag{27}$$

$$\tilde{\gamma}^j(\tilde{d}) \equiv \max_{\eta,\alpha}\left(\frac{\pi_j}{1-\psi}\hat{u}_i(\alpha) - \frac{\pi_i}{1-\psi}\hat{u}_j(\alpha) + \frac{\psi}{1-\psi}\hat{\eta}(\alpha)\right) \tag{28}$$

$$\text{s.t.}\quad \begin{cases} \eta : \Theta \to [-\tilde{d}, 0] \text{ and } \hat{\eta} = \sum_{\theta\in\Theta} f(\theta|\cdot)\eta(\theta), \\ \alpha \in \hat{\Delta}\mathcal{A} \text{ is a Nash equilibrium} \\ \quad \text{of } \langle \mathcal{A}_i \times \mathcal{A}_j, (\hat{u}_i, \hat{u}_j) + \boldsymbol{\delta} * (\hat{\eta}, -\hat{\eta})\rangle. \end{cases}$$

(ii) *The* $\operatorname{level}(\tilde{W}^*)$ *is equal to*

$$\begin{aligned} &\frac{1-\psi}{(1-\delta_1)(1-\delta_2)}\Big(\delta_1\tilde{\gamma}^2\big(\operatorname{span}(\tilde{W}^*)\big) + \delta_2\tilde{\gamma}^1\big(\operatorname{span}(\tilde{W}^*)\big) \\ &\qquad - \delta_2 \operatorname{span}(\tilde{W}^*) + \chi\big(\operatorname{span}(\tilde{W}^*)\big)\Big), \end{aligned} \tag{29}$$

where

$$\chi(\tilde{d}) = \max_{\eta,\alpha} \hat{u}_1(\alpha) + \hat{u}_2(\alpha) + (\delta_1 - \delta_2)\hat{\eta}(\alpha) \tag{30}$$

$$\text{s.t.}\quad \begin{cases} \eta : \Theta \to [-\tilde{d}, 0] \text{ and } \hat{\eta} = \sum_{\theta\in\Theta} f(\theta|\cdot)\eta(\theta), \\ \alpha \in \hat{\Delta}\mathcal{A} \text{ is a Nash equilibrium of } \langle \mathcal{A}, \hat{u} + \boldsymbol{\delta} * (\hat{\eta}, -\hat{\eta})\rangle. \end{cases}$$

(iii) *The endpoints* $\tilde{W}^*$ *are*

$$\tilde{z}^1 = (-1, 1)\tilde{\gamma}^1\big(\mathrm{span}\big(\tilde{W}^*\big)\big) + \left(\frac{1-\delta_2}{1-\psi}, \frac{1-\delta_1}{1-\psi}\right) * \pi\,\mathrm{level}\big(\tilde{W}^*\big), \tag{31}$$

$$\tilde{z}^2 = (1, -1)\tilde{\gamma}^2\big(\mathrm{span}\big(\tilde{W}^*\big)\big) + \left(\frac{1-\delta_2}{1-\psi}, \frac{1-\delta_1}{1-\psi}\right) * \pi\,\mathrm{level}\big(\tilde{W}^*\big). \tag{32}$$

The remainder of this section proves Theorem 8, Theorem 11, and Corollary 2. Let $\mathcal{X}$ denote the set of compact subsets of $\mathbb{R}^n$ and let $\mathcal{X}_0$ denote the set of compact subsets of $\mathbb{R}^n_0$. For any set $X \in \mathcal{X}$ and any point $x' \in \mathbb{R}^n$, let the sum be defined by $X + x' \equiv \{x + x' | x \in X\}$. For every $X \in \mathcal{X}$, define $\hat{B}(X) \equiv \mathrm{co}\, B(C(X), D(X))$, $L(X) \equiv \max_{x\in X} \sum_{i=1}^n x_i$, and $\tilde{B}(X) \equiv \hat{B}(X) - \pi L(\hat{B}(X))$. The function $\tilde{B}$ normalizes the output of $\hat{B}$ so that points in the resulting sets have a joint value of zero. The angle by which this normalization takes place (in the direction of π) is critical for the analysis below.

LEMMA 13: *Function* $\hat{B}$ *maps* $\mathcal{X}$ *to* $\mathcal{X}$ *and function* $\tilde{B}$ *maps* $\mathcal{X}$ *to* $\mathcal{X}_0$. *For every* $\tilde{v} \in \hat{B}(X)$, *it is the case that* $\sum_{j=1}^n \tilde{v}_j = L(\hat{B}(X))$. *Furthermore*, *for any* $X \in \mathcal{X}$ *and* $x \in \mathbb{R}^n$, $\hat{B}(X + x) = \hat{B}(X) + \boldsymbol{\delta} * x$ *and* $L(X + x) = L(X) + L(\{x\})$.

PROOF: By upper hemicontinuity of the Nash equilibrium correspondence, the operators C and D preserve compactness. The bargaining outcome is clearly continuous in the disagreement point and maximal joint value. Furthermore, $\tilde{B}(X)$ is a linear transformation of $\hat{B}(X)$ that makes every point balanced (joint value of zero). These facts imply the first part of the lemma. The second part follows from transferable utility and the bargaining solution (as in Lemma 1). Regarding the third part, note that adding x to every point in X merely shifts the set of continuation values from the next period. Referring to Eq. (13), this is equivalent to replacing $\tilde{g}(\theta)$ with $\tilde{g}(\theta) + x$. Thus, the set of values that can be supported from the current period uniformly shifts by $\boldsymbol{\delta} * x$. Consequently, the set of bargaining outcomes likewise shifts. Finally, the level clearly changes as indicated. *Q.E.D.*

Hereinafter, all sets that we consider are understood to be compact subsets of $\mathbb{R}^n$. To establish the existence of a dominant BSG set, we first characterize the BSG sets—the fixed points of $\hat{B}$. To identify and compare BSG sets, we work with the function $\tilde{B}$. We start by demonstrating a relation between the fixed points of $\hat{B}$ and $\tilde{B}$. For each player i, define $\varphi_i \equiv \pi_i/(1 - \delta_i)$, write $\varphi = (\varphi_1, \varphi_2, \dots, \varphi_n)$, and let $\Phi \equiv \sum_{j=1}^n \varphi_j$. Lemma 14 provides a linear relationship between the fixed points of $\hat{B}$ and $\tilde{B}$ in the direction φ.[32]

[32]This relationship does not hold for other sets in general.

LEMMA 14: *If* $\tilde{W} \in \mathcal{X}$ *and* $X = \tilde{W} - \frac{\varphi}{\Phi} L(\tilde{W})$, *then* $\tilde{W} = \hat{B}(\tilde{W})$ *implies* $X = \tilde{B}(X)$. *If* $X \in \mathcal{X}_0$ *and* $\tilde{W} = X + \varphi L(\hat{B}(X))$, *then* $X = \tilde{B}(X)$ *implies* $\tilde{W} = \hat{B}(\tilde{W})$.

PROOF: To prove the first part of the lemma, we start with some algebraic steps:

$$\begin{aligned} \tilde{B}(X) &= \hat{B}(X) - \pi L\big(\hat{B}(X)\big) \\ &= \tilde{W} - \boldsymbol{\delta} * \frac{\varphi}{\Phi} L(\tilde{W}) - \pi L\left(\tilde{W} - \boldsymbol{\delta} * \frac{\varphi}{\Phi} L(\tilde{W})\right) \\ &= \tilde{W} - \boldsymbol{\delta} * \frac{\varphi}{\Phi} L(\tilde{W}) - \pi L(\tilde{W})\left(1 - \boldsymbol{\delta} \cdot \frac{\varphi}{\Phi}\right) \\ &= \tilde{W} - \left(\pi + (\mathbf{1} - \pi)\boldsymbol{\delta} * \frac{\varphi}{\Phi}\right) L(\tilde{W}). \end{aligned} \tag{33}$$

The second line uses the property of $\hat{B}$ from Lemma 13 and that $V = \hat{B}(V)$. The third line uses the property of L from Lemma 13. Note that

$$\Phi - \boldsymbol{\delta} \cdot \varphi = \sum_{j=1}^{n} \varphi_j (1 - \delta_j) = \sum_{j=1}^{n} \frac{\pi_j}{1 - \delta_j}(1 - \delta_j) = 1. \tag{34}$$

Thus,

$$\tilde{B}(X) = \tilde{W} - \boldsymbol{\delta} * \frac{\varphi}{\Phi} L(\tilde{W}) - \pi \frac{1}{\Phi} L(\tilde{W}) = \tilde{W} - (\boldsymbol{\delta} * \varphi + \pi)\frac{1}{\Phi} L(\tilde{W}). \tag{35}$$

It can be verified that $\delta_i \varphi_i + \pi_i = \varphi_i$, which means that $\boldsymbol{\delta} * \varphi + \pi = \varphi$. Therefore, $\tilde{B}(X) = \tilde{W} - \frac{\varphi}{\Phi} L(\tilde{W}) = X$.

To prove the second part of the lemma, we perform the algebraic steps

$$\begin{aligned} \hat{B}(\tilde{W}) &= \hat{B}\big(X + \varphi L\big(\hat{B}(X)\big)\big) \\ &= \hat{B}(X) + \boldsymbol{\delta} * \varphi L\big(\hat{B}(X)\big) \\ &= \hat{B}(X) - \pi L\big(\hat{B}(X)\big) + \pi L\big(\hat{B}(X)\big) + \boldsymbol{\delta} * \varphi L\big(\hat{B}(X)\big) \\ &= \tilde{B}(X) + (\pi + \boldsymbol{\delta} * \varphi) L\big(\hat{B}(X)\big) = X + \varphi L\big(\hat{B}(X)\big) = \tilde{W}. \end{aligned} \tag{36}$$

The second line uses the property of $\hat{B}$ from Lemma 13 and the third line uses the definition of $\tilde{B}$. The fourth line uses the assumption that $\tilde{B}(X) = X$ and that $\boldsymbol{\delta} * \varphi + \pi = \varphi$, which we showed above. *Q.E.D.*

We next show by construction that $\tilde{B}$ has a dominant fixed point. We start by identifying some properties of $\tilde{B}$.

LEMMA 15: *The function $\tilde{B}$ is monotone*: *for every $X, X' \in \mathcal{X}$, $X \subset X'$ implies $\tilde{B}(X) \subset \tilde{B}(X')$. Furthermore, $\tilde{B}$ is continuous on decreasing sequences*: *for every sequence $\{X^k\} \subset \mathcal{X}$ with $X^{k+1} \subset X^k$ for all k, if X^k converges to X in the Hausdorff metric, then $\tilde{B}(X^k)$ converges to $\tilde{B}(X)$.*

PROOF: To prove the first part of the lemma, take sets $X, X' \in \mathcal{X}$ such that $X \subset X'$. Clearly D is monotone, so $D(X) \subset D(X')$. By construction, we know that for each point $\tilde{w} \in B(C(X), D(X))$, there is an element $\underline{\tilde{w}} \in D(X)$ such that $\tilde{w} = \underline{\tilde{w}} + \pi(L(D(X)) - \mathbf{1} \cdot \underline{\tilde{w}})$. Using the same disagreement point and choosing $\tilde{w}' = \underline{\tilde{w}} + \pi(L(D(X')) - \mathbf{1} \cdot \underline{\tilde{w}})$, we have that $\tilde{w}' \in B(C(X'), D(X'))$. Thus, $\tilde{w} \in \hat{B}(X)$, $\tilde{w}' \in \hat{B}(X')$, and these points lie on the same line in direction π. Recall that $\tilde{B}$ normalizes by subtracting a vector proportional to π, so that resulting points have joint value of zero. This means that $\tilde{w} - \pi L(\hat{B}(X)) = \tilde{w}' - \pi L(\hat{B}(X'))$, which proves $\tilde{B}(X) \subset \tilde{B}(X')$.

To prove the second part of the lemma, consider any decreasing sequence $\{X^k\} \subset \mathcal{X}$ such that X^k converges to X for some $X \in \mathcal{X}$. Because the Nash equilibrium correspondence is upper hemicontinuous, we know that $\lim_{k\to\infty} D(X^k) \subset D(X)$. Since $\{X^k\}$ is decreasing, we have that $X \subset X^k$ for all k. Because D is monotone, $D(X) \subset D(X^k)$ holds, and this implies that $D(X) \subset \lim_{k\to\infty} D(X^k)$. Thus, $D(X^k)$ converges to $D(X)$. By the same reasoning, $C(X^k)$ converges to $C(X)$. The function B is continuous as described in the proof of Lemma 13. Thus, $\hat{B}(X^k)$ converges to $\hat{B}(X)$ and so $\tilde{B}(X^k)$ converges to $\tilde{B}(X)$. *Q.E.D.*

The next lemma follows from the fact that weakly more action profiles in the stage game can be enforced for larger sets of continuation values.

LEMMA 16: *For $X, X' \in \mathcal{X}_0$, $X \subset X'$ implies that $L(\hat{B}(X)) \leq L(\hat{B}(X'))$.*

PROOF OF THEOREM 8: To construct a dominant fixed point of $\tilde{B}$, we first construct a large element of $\mathcal{X}_0$ that is guaranteed to be a superset of any fixed point. Since Θ and $\mathcal{A}$ are finite, there exists $K \in \mathbb{R}_+$ such that all stage-game payoffs are bounded below by $-K(1 - \max_j \delta_j)$ and above by $K(1 - \max_j \delta_j)$. Let $X^1 \equiv \{x \in \mathbb{R}^n_0 \mid -K \leq x \leq K \text{ for all } i\}$. Then every fixed point of $\tilde{B}$ is a subset of X^1 and that $\tilde{B}(X^1) \subset X^1$. Define the sequence $\{X^k\}$ inductively by $X^{k+1} \equiv \tilde{B}(X^k)$, for all $k > 1$. Since $\tilde{B}$ is monotone, this sequence is decreasing in the set inclusion order. Furthermore, $\{X^k\} \subset \mathcal{X}_0$. Since every decreasing sequence of compact sets in a Euclidean space converges, there exists $X^* \in \mathcal{X}_0$ to which X^k converges; moreover, by Tarski's fixed-point theorem, X^* is the largest fixed point of $\tilde{B}$.

Lemma 15 implies that $X^* = \tilde{B}(X^*)$. To see this, note that $X^* \subset X^{k+1} = \tilde{B}(X^k)$ for all k. Because $\tilde{B}$ is continuous on decreasing sequences, $\tilde{B}(X^k)$ converges to $\tilde{B}(X^*)$ and so we have that $X^* \subset \tilde{B}(X^*)$. In addition, because

$\tilde{B}$ is monotone and $X^* \subset X^k$, we have $\tilde{B}(X^*) \subset \tilde{B}(X^k) = X^{k+1}$ for all k. That X^{k+1} converges to X^* then implies that $\tilde{B}(X^*) \subset X^*$.

Next we argue that every fixed point of $\tilde{B}$ is a subset of X^*. Suppose that this were not the case, so that there is a set $X \in \mathcal{X}_0$ such that $X = \tilde{B}(X)$ but $X \not\subset X^*$. Then we can find a positive integer K such that $X \subset X^k$ for all $k \leq K$, but $X \not\subset X^{K+1}$. This violates monotonicity of $\tilde{B}$, which requires $X = \tilde{B}(X) \subset \tilde{B}(X^K) = X^{K+1}$.

Thus, we have established that X^* is a fixed point of $\tilde{B}$ and every other fixed point of $\tilde{B}$ is contained in X^*. Define $\tilde{W}^* \equiv X^* + \varphi L(\hat{B}(X^*))$. We finish the proof by showing that $\tilde{W}^*$ is a BSG set for the game and it dominates every other BSG set. That $\tilde{W}^*$ is an BSG set follows immediately from Lemma 14. For the second step, consider any other BSG set $\tilde{W}$ and define $X \equiv \tilde{W} - \frac{\varphi}{\Phi} L(\tilde{W})$. From Lemma 14 we know that X is a fixed point of $\tilde{B}$. We also know that $X \subset X^*$.

From the relationship between fixed points of $\underline{w}^1$ and $\tilde{B}$, we know that $\tilde{W} = X + \varphi L(\underline{w}^1(X))$. Take any $\tilde{v} \in \tilde{W}$ and let $\tilde{x} \in X$ be such that $\tilde{v} = \tilde{x} + rL(\underline{w}^1(X))$. Since $X \subset X^*$, we have that $\tilde{x} \in X^*$ and thus $\tilde{v}' \equiv \tilde{x} + \varphi L(\underline{w}^1(X^*)) \in \tilde{W}$. Comparing $\tilde{v}$ and $\tilde{v}'$, we see that $\tilde{v}' - \tilde{v} = \varphi(L(\underline{w}^1(X^*)) - L(\underline{w}^1(X)))$. From Lemma 16, we know that $L(\underline{w}^1(X^*)) \geq L(\underline{w}^1(X))$. In addition, note that $\varphi_i \geq 0$ for all i. These facts imply that $\tilde{v}' \geq \tilde{v}$ (that is, $\tilde{v}'_i \geq \tilde{v}_i$ for every player i), which proves that $\tilde{W}^*$ dominates $\tilde{W}$. *Q.E.D.*

PROOF OF THEOREM 11: We first work through the construction of $\tilde{z}^2$, which is player 1's most preferred point in $\tilde{W}^*$. Everything is analogous for $\tilde{z}^1$. In this environment, Eq. (21) becomes

$$\tilde{z}_1^2 = \max_{\underline{\tilde{w}}, \tilde{g}, \alpha} \bigl(\pi_2 \underline{\tilde{w}}_1 - \pi_1 \underline{\tilde{w}}_2 + \pi_1 \operatorname{level}\bigl(\tilde{W}^*\bigr)\bigr) \tag{37}$$

$$\text{s.t.} \quad \begin{cases} \underline{\tilde{w}} = \displaystyle\sum_{\theta \in \Theta} f(\theta|\alpha)\bigl(u(\alpha, \theta) + \boldsymbol{\delta} * \tilde{g}(\theta)\bigr), \\ \tilde{g} : \Theta \to \tilde{W}^* \text{ enforces } \alpha. \end{cases}$$

Define $\eta(\theta) \equiv \tilde{g}_1(\theta) - \tilde{z}_1^2$ and $\hat{u}_i(\alpha) \equiv \sum_{\theta \in \Theta} f(\theta|\alpha) u_i(\alpha_i, \theta)$. After rearranging terms as before, the optimization problem of Eq. (22) becomes

$$\begin{aligned} \tilde{z}_1^2 = \max_{\eta, \alpha} \Bigl(&\bigl(\pi_2 \hat{u}_1(\alpha) - \pi_1 \hat{u}_2(\alpha)\bigr) + \bigl(\pi_2 \delta_1 \tilde{z}_1^2 - \pi_1 \delta_2 \tilde{z}_2^2\bigr) \\ &+ (\pi_2 \delta_1 + \pi_1 \delta_2) \hat{\eta}(\alpha) + \pi_1 \operatorname{level}\bigl(\tilde{W}^*\bigr)\Bigr) \end{aligned} \tag{38}$$

$$\text{s.t.} \quad \begin{cases} \eta : \Theta \to \bigl[\tilde{z}_1^1 - \tilde{z}_1^2, 0\bigr], \text{ with } \hat{\eta}(\alpha) \equiv \displaystyle\sum_{\theta \in \Theta} f(\theta|\alpha)\eta(\theta), \\ \alpha \in \hat{\Delta}\mathcal{A} \text{ is a Nash equilibrium} \\ \quad \text{of } \bigl\langle \mathcal{A}, \hat{u} + \boldsymbol{\delta} * \bigl((\hat{\eta}, -\hat{\eta}) + \tilde{z}^2\bigr)\bigr\rangle. \end{cases}$$

Substituting in ψ, using $\tilde{z}_1^2 + \tilde{z}_2^2 = \text{level}(\tilde{W}^*)$, and rearranging terms yields

$$\tilde{z}_1^2 = \frac{\pi_1(1-\delta_2)}{1-\psi}\,\text{level}(\tilde{W}^*) + \tilde{\gamma}^2(\text{span}(\tilde{W}^*)), \tag{39}$$

where γ^2 is defined in Theorem 11. Similar calculations yield

$$\tilde{z}_2^1 = \frac{\pi_2(1-\delta_1)}{1-\psi}\,\text{level}(\tilde{W}^*) + \tilde{\gamma}^1(\text{span}(\tilde{W}^*)). \tag{40}$$

Summing these expressions, we have

$$\tilde{z}_2^1 + \tilde{z}_1^2 = \text{level}(\tilde{W}^*) + \tilde{\gamma}^1(\text{span}(\tilde{W}^*)) + \tilde{\gamma}^2(\text{span}(\tilde{W}^*)). \tag{41}$$

Substituting $\tilde{z}_2^1 + \tilde{z}_1^1 = \text{level}(\tilde{W}^*)$ for $\tilde{z}_2^1$ yields $\text{span}(\tilde{W}^*) = \tilde{\gamma}^1(\text{span}(\tilde{W}^*)) + \tilde{\gamma}^2(\text{span}(\tilde{W}^*))$, so we conclude that $\text{span}(\tilde{W}^*)$ is a fixed point of $\tilde{\Gamma}$ as in the basic model.

The construction of $\tilde{W}^*$ proceeds as in the basic model. We first find the maximal fixed point of $\tilde{\Gamma}$. We then have to calculate $\text{level}(\tilde{W}^*)$, which is a bit more involved than in the basic model because it is the infinite sum of discounted payoffs where the players have different discount factors. Note that the level satisfies

$$\begin{aligned} \text{level}(\tilde{W}^*) &= \max_{\tilde{g},\alpha}\Bigl(\hat{u}_1(\alpha) + \hat{u}_2(\alpha) + \sum_{\theta\in\Theta} f(\theta|\alpha)\boldsymbol{\delta}\cdot\tilde{g}(\theta)\Bigr) \\ &\quad \text{s.t.}\quad \tilde{g}:\Theta\to\tilde{W}^* \text{ enforces } \alpha, \end{aligned} \tag{42}$$

because the objective function here is the joint value from the current period and the constraint requires that continuation values in the following period be chosen from $\tilde{W}^*$.

With the same steps taken above, we rewrite this maximization problem by substituting for $\tilde{g}$ using the function η, where we have $\tilde{g}_1(\theta) = \eta(\theta) + \tilde{z}_1^2$ and $\tilde{g}_2(\theta) = \tilde{z}_2^1 - \text{span}(\tilde{W}^*) - \eta(\theta)$. This yields

$$\text{level}(\tilde{W}^*) = \delta_1\tilde{z}_1^2 + \delta_2\tilde{z}_2^1 - \delta_2\,\text{span}(\tilde{W}^*) + \chi(\text{span}(\tilde{W}^*)), \tag{43}$$

where

$$\begin{aligned} \chi(\tilde{d}) &= \max_{\eta,\alpha}\hat{u}_1(\alpha) + \hat{u}_2(\alpha) + (\delta_1-\delta_2)\hat{\eta}(\alpha) \\ &\quad \text{s.t.}\quad \begin{cases} \eta:\Theta\to[-\tilde{d},0], \text{ with } \hat{\eta}(\alpha)\equiv\sum_{\theta\in\Theta} f(\theta|\alpha)\eta(\theta), \\ \alpha\in\hat{\Delta}\mathcal{A} \text{ is a Nash equilibrium of } \langle\mathcal{A}, \hat{u}+\boldsymbol{\delta}*(\hat{\eta},-\hat{\eta})\rangle. \end{cases} \end{aligned} \tag{44}$$

We can use Eq. (39) and Eq. (40) to substitute for $\tilde{z}_1^2$ and $\tilde{z}_2^1$ in Eq. (43). After solving for level$(\tilde{W}^*)$ and simplifying, we obtain

$$\text{level}(\tilde{W}^*) = \frac{1-\psi}{(1-\delta_1)(1-\delta_2)}\big(\delta_1\tilde{\gamma}^2(\text{span}(\tilde{W}^*)) + \delta_2\tilde{\gamma}^1(\text{span}(\tilde{W}^*)) - \delta_2\,\text{span}(\tilde{W}^*) + \chi(\text{span}(\tilde{W}^*))\big). \tag{45}$$

The foregoing argument proves the first and second parts of the theorem. The third part follows from Eqs. (39) and (40), and that the joint value is level$(\tilde{W}^*)$. *Q.E.D.*

PROOF OF COROLLARY 2: Let $\tilde{W}^*(\delta)$ be the contractual equilibrium set when $\delta_1 = \delta_2 = \delta$. Letting $\delta_1 = \delta + \varepsilon$, $\delta_2 = \delta - \varepsilon$, $\pi_1 = \frac{1}{2} + \hat{\varepsilon}$, and $\pi_2 = \frac{1}{2} - \hat{\varepsilon}$, we compute that

$$\left.\frac{\partial \bar{z}_1}{\partial \varepsilon}\right|_{\varepsilon=0} = \frac{-\tilde{\gamma}^1(\text{span}(\tilde{W}^*)) + \tilde{\gamma}^2(\text{span}(\tilde{W}^*)) + (2 + 6\hat{\varepsilon} + 4\hat{\varepsilon}^2)\,\text{level}(\tilde{W}^*)}{2(1-\delta)^2}. \tag{46}$$

If the stage game is symmetric and bargaining power is equal, then $\tilde{\gamma}^1(\text{span}(\tilde{W}^*)) = \tilde{\gamma}^2(\text{span}(\tilde{W}^*))$ and $\hat{\varepsilon} = 0$, so player 1's average payoff across the two states is strictly increasing in ε. By symmetry, player 2's average payoff across the two states is strictly decreasing in ε.

We also compute that

$$\left.\frac{\partial \bar{z}}{\partial \varepsilon}\right|_{\varepsilon=0} = \frac{-\tilde{\gamma}^1(\text{span}(\tilde{W}^*)) + \tilde{\gamma}^2(\text{span}(\tilde{W}^*)) + 6\hat{\varepsilon}\,\text{level}(\tilde{W}^*)}{(1-\delta)^2}, \tag{47}$$

$$\left.\frac{\partial^2 \bar{z}}{\partial \varepsilon^2}\right|_{\varepsilon=0} = \frac{4(1 + 6\hat{\varepsilon}^2)\,\text{level}(\tilde{W}^*)}{(1-\delta)^3}. \tag{48}$$

If the stage game is symmetric and bargaining power is equal, then $\varepsilon = 0$ is a strict local minimizer of the average welfare across the two states. *Q.E.D.*

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Manuscript received October, 2011; final revision received March, 2013.