When to behave badly and when to behave well under disagreement

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Abstract

In a repeated principal-agent problem in which the agent has private information about her i.i.d. cost of effort (à la Levin 2003), we analyze relational contracts that the parties can renegotiate in a way that respects their relative bargaining power. We show that if a disagreement arises in a state in which she was to be rewarded, then it is optimal for the agent to destroy surplus, exerting costly effort to hurt the principal. In such an event, her counter-productive effort is optimally constant regardless of her effort cost, the principal does not fire her, and both parties anticipate agreeing to reward the agent in the next period. In contrast, on the equilibrium path as well as under disagreement in a state in which the agent was to be punished, the agent exerts productive effort that is decreasing in her effort cost.

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1 Introduction

If the agent in a principal-agent relationship has private information about the returns to effort, then—even if effort is fully contractible and can be externally enforced—the parties must employ "relational" incentives to self-enforce their activities. In a celebrated paper, Levin (2003) showed how to construct optimal relational contracts with hidden information, demonstrating that such contracts are stationary and satisfy a basic notion of renegotiation proofness.

In this paper we consider the same model, but apply a solution concept for relational contracts, *contractual equilibrium* (Miller and Watson 2013), that allows the parties to renegotiate in a way that recognizes their relative bargaining power. We show that the optimal relational contract, while stationary on the equilibrium path, enforces surplus-destroying, counter-productive behavior at some histories off the equilibrium path. Specifically, the agent is called upon to destroy surplus when both (1) the principal and the agent fail to reach agreement while attempting to renegotiate, and (2) the agent has been promised a high payoff due to past efforts. When destroying surplus, the agent optimally exerts the same counter-productive effort regardless of her current effort cost. This contrasts to the equilibrium path, whereon her productive effort schedule, as a function of her effort cost, is typically partially separating.

Destroying surplus is costly to both the principal and the agent—envision a worker picketing in front of her employer's storefront, but not being fired—but it serves an important function. By reducing payoffs under disagreement, such counterproductive effort increases the gains to agreement, and it shifts the disagreement point in a way that disadvantages the principal relative to the agent. Counterproductive effort, and the concomitant shift in the agent's bargaining position, occurs only after a disagreement in a state of the world in which the agent is supposed to be rewarded. When the agent is being punished, she is supposed to exert the same effort regardless of whether she and the principal reach agreement, so there is no surplus to bargain over. The optimal contract thereby maximizes the difference between the agent's agreement payoff in reward states versus punishment states, providing the highest-powered incentives on the equilibrium path.

Since off-path surplus-destroying behavior creates incentives through the channel of the agent's bargaining power, the principal ex ante prefers an intermediate level of bargaining power. Our interpretation is that at the outset of the relationship, though the principal may initially possess all the bargaining power, he would like to "sell" some bargaining power to the agent in exchange for a lower starting wage. Whether bargaining power can really be sold depends on the institutional environment. For instance, labor laws may govern the bargaining process; even if not, the parties may still be able to write certain forms of arbitration into their formal contract.

Typically, the agent's optimal effort schedule along the equilibrium path is partially separating—

it is better for her to exert more effort when it is less costly to do so, if the incentive power is available to enforce partial separation. In contrast, when the agent is to be rewarded but an off-path disagreement arises, it is optimal for the agent's counter-productive effort to be insensitive to her effort cost. Intuitively, the point of destroying surplus off path is to improve the agent's best bargaining outcome. Unfortunately, since she cannot receive more than her maximum continuation reward, while increasing the variance in her information rent from separating might enable her to destroy more surplus on average, it would also decrease her expected continuation value. If there is no separation among her types, then they can all earn the maximum continuation reward, increasing her disagreement payoff at the expense of the principal.

In this model, we show that a contractual equilibrium is the solution to a fixed point problem relating two continuous optimal control problems. The first of these problems, representing the agent's equilibrium-path effort schedule, has one control and two states, and is similar to the problem analyzed by Levin (2003). The second of these problems, representing the agent's counterproductive effort schedule under disagreement in her reward state, has one control and four states, and is substantially more complex. Nonetheless, we are able to show that the solution to the second problem cannot be partially separating. At this time we are unable to rule out the possibility of full separation, but we can show that any fully separating effort schedule must be the solution to an overdetermined system of equations. Hence we strongly conjecture that only full pooling can arise; i.e., all agent types select the same level of counterproductive effort.

Our analysis and conclusions differ from those of Levin (2003) because of our different choice of equilibrium refinement. Levin finds an optimal perfect public equilibrium subject to the renegotiation proofness notion of "strong optimality," which requires that expected payoffs at the start of each period must always be on the Pareto frontier of what is attainable in any equilibrium. Levin constructs these equilibria by requiring a player who deviates to give all the surplus to the other player. Specifically, if either player deviates in the bargaining phase (e.g., the principal makes a deviant offer or the agent rejects the equilibrium offer), then they take their outside options for one period and then continue by giving all the surplus to the player who did not deviate. It follows that in such an equilibrium the entire economic surplus can be used to provide incentives for the agent. "Strongly optimal" equilibria do not entertain the possibility of bargaining power, since the net surplus at stake in the bargaining phase (one period of optimal production if they agree minus one period of outside options if they disagree) is allocated endogenously as a function of the history rather than according to a bargaining protocol.[1]

In contrast, contractual equilibrium assumes that bargaining power is fixed exogenously—e.g.,

[1]For concreteness, Levin assumes that the principal makes a take-it-or-leave-it offer, but notes that this privilege confers no bargaining power, and that his results would not change if the agent instead got to make the offers.

via the bargaining protocol that the parties must follow when negotiating. Because the principal retains his bargaining power even when the agent is being rewarded, not all the surplus can be used to incentivize the agent. Contractual equilibrium also imposes no ad hoc constraints on behavior under disagreement. Instead, it allows the parties to plan their behavior under potential future disagreements so as to maximize the agent's incentives under agreement. Surplus destruction under disagreement in the agent's reward states improves matters by increasing the fraction of the equilibrium-path surplus that can be devoted to the agent's incentives.

Miller and Watson (2013) briefly analyzed a simpler principal-agent relationship with moral hazard, also based on a model from Levin (2003). They showed that in the moral hazard setting the agent's incentives are increasing in her bargaining power; we extend that conclusion to the more challenging setting of hidden information. They also showed that if counterproductive agent effort is formally contractible and legally enforceable, then the optimal formal contract monetarily rewards the agent for engaging in counterproductive effort. Here we demonstrate that counterproductive effort need not be legally enforceable to be helpful.

2 The model

A principal P and an agent A interact in a repeated game, in discrete time with an infinite horizon. They share a common discount factor $\delta \in (0, 1)$. The stage game has the following extensive form structure within any period t:

1. First, the principal and the agent engage in bargaining, which we describe later when introducing contractual equilibrium. The bargaining process is parameterized by the parties' *bargaining powers*: $\pi_A \in [0, 1]$ for the agent, and $\pi_P = 1 - \pi_A$ for the principal. The outcome of the bargaining process is either "disagreement" with no monetary transfer, or "agreement" with an immediate monetary transfer and an informal contract specifying a strategy profile for continuation play.

2. Before continuing, either party can unilaterally opt to terminate the relationship permanently (i.e., the principal can opt to fire the agent, and the agent can opt to quit). If either party terminates the relationship, then both parties receive their outside options in the current and all future periods.

3. Next, the agent privately observes her cost type θ, which is drawn i.i.d. (i.e., regardless of the history) from a probability distribution represented by a PDF ϕ on $\Theta \equiv [\underline{\theta}, \overline{\theta}]$.

4. If neither party terminated the relationship, the agent chooses her effort $e \in \mathbb{R}_+$. Her cost of effort is $c(e, \theta)$.[2]

5. Given her effort, the agent decides how much revenue y should accrue to the principal. The revenue must satisfy $-b(e) \leq y \leq g(e)$; that is, g and b define a production possibilities frontier in the product space of costs to the agent and revenues to the principal.

Other than the agent's type θ, all other actions and outcomes are observed by both parties. If neither player terminates the relationship, the stage game payoffs are $u_A = \tau - c(e, \theta)$ for the agent and $u_P = -\tau + y$ for the principal, where τ is the net monetary payment from the principal to the agent in the bargaining phase. If either player terminates the relationship, the stage game payoffs in that period are $u_A = \tau + \overline{u}_A$ and $u_P = -\tau + \overline{u}_P$, and then in all future periods they receive constant per-period payoffs of $\overline{u}_A$ and $\overline{u}_P$.

Remark 1. Although our assumptions on the timing of monetary transfers differ from those of Levin (2003), the two models are essentially equivalent in this regard. Levin assumes that the agreement at the start of the period specifies a legally enforced "fixed salary" that the principal must pay at the end of the period, but this is identical to paying immediately upon agreement. Similarly, Levin assumes that the parties may pay each other voluntary non-negative "bonuses" at the end of the period. Parties do not want to renege on these bonus payments because a reneger faces a punishment at the start of the next period. So no substantive change is imposed by assuming that the bonus payment is delayed until the start of the next period: although the monetary amount of the payment must be increased by $1/\delta$, the punishment that enforces it can be implemented immediately, rather than with delay, and is therefore strengthened by the same factor.

We make the following standard assumptions on the primitives (see Levin 2003):

Assumption 1. *The functions c, g, b, and ϕ satisfy the following:*

1. *The cost function c is smooth, with $c(0, \theta) = 0$.*

2. *For all $e \in (0, \infty)$ and all $\theta \in (\underline{\theta}, \overline{\theta})$, $c_e(e, \theta) > 0$, $c_{ee}(e, \theta) > 0$, $c_\theta(e, \theta) \geq 0$, $c_{e\theta}(e, \theta) > 0$, $c_{e\theta e}(e, \theta) \geq 0$, and $c_{e\theta\theta}(e, \theta) \geq 0$.*

3. *The production frontier functions g and b are smooth, with $g(0) = b(0) = 0$.*

4. *For all $e \in (0, \infty)$, $g_e(e) > 0$, $g_{ee}(e) < 0$, $b_e(e) > 0$, and $b_{ee}(e) < 0$.*

5. *For each $\theta \in \Theta$, $g(e) - c(e, \theta)$ has an interior maximizer $e^{\text{FB}}(\theta) \in (0, \infty)$.*

[2] Levin assumes effort is bounded, but in light of his assumption that first best effort is interior, the bound is not substantive.

6. *The distribution of types has full support on* Θ.

We also make the following additional assumption on combinations of primitives:

Assumption 2. *For all* $e \in (0, \infty)$ *and all* $\theta \in (\underline{\theta}, \overline{\theta})$,

1. $\frac{\partial}{\partial \theta}\big(g_e(e) - c_e(e, \theta)\big)\phi(\theta) < 0$, *and*

2. $\frac{\partial}{\partial \theta}\big(b_e(e) + c_e(e, \theta)\big)\phi(\theta) > 0$.

The first part of this assumption is weaker than Levin's assumption that the cumulative distribution of types is concave; the second part is mathematically similar but prevents the cumulative distribution from being "too concave" in the context of c and b. As an example, given Assumption 1, Assumption 2 is satisfied if the distribution of types is uniform, or, in the context of c, b, and g, not "too far" from uniform.

Finally, we make the following restrictive third derivative assumption:

Assumption 3. *For all* $e \in (0, \infty)$ *and all* $\theta \in (\underline{\theta}, \overline{\theta})$, $c_{e\theta\theta}(e, \theta) = 0$.

This assumption is used only in Lemma 2, where we use it in combination with Assumption 2 to rule out the possibility of partial separation when the agent engages in surplus destruction. The proof of Lemma 2 makes clear that this is not a knife-edge conclusion; in the context of any given b and ϕ, it is substantially stronger than needed. What is needed is that $c_{e\theta}$ not increase in θ too fast relative to $\big(b_e + c_e\big)\phi$. Unfortunately, what would constitute "too fast" is endogenously determined, so it is difficult to propose an appropriate weakening of this assumption.

3 The optimal relational contract

Miller and Watson (2013) show that a contractual equilibrium strategy profile in a two-player game is, without loss of generality, defined by behavior in and transitions among agreement and disagreement in both in the agent's reward state (H) and punishment state (L). Such a strategy profile constitutes a contractual equilibrium if it maximizes the parties' welfare in the agreement states, subject to:

- the agent's incentive compatibility—choosing the right effort and the right revenue as a function of her type;
- both parties' individual rationality—never strictly preferring to terminate the relationship;
- and the bargaining operator—splitting the difference in welfare between agreement and disagreement in proportion to their bargaining powers.

Though the bargaining operator embodies the "cooperative" notion of Nash bargaining, Miller and Watson (2013) proved that this hybrid approach in which a cooperative bargaining stage is embedded in an otherwise noncooperative game is equivalent to a fully non-cooperative approach with cheap-talk bargaining, under several equilibrium selection axioms on the interpretation of cheap-talk messages. Justified by their results, in this paper we exclusively employ the hybrid approach.

We can make three simplifications immediately, without loss. First, since the agent can freely choose any revenue $y \in [-b(e), g(e)]$, incentive compatibility of her revenue choice conditional on her effort choice is trivially guaranteed by simply transitioning to her punishment state for any violation. Second, under agreement the agent's revenue choice conditional on her effort must always be $y = g(e)$—otherwise she could increase welfare by simply choosing higher revenue, without any change in incentive provision. Third, the agent's effort as a function of her type must be identical in both states under agreement—otherwise welfare would not be maximized on the equilibrium path.

For now, we assert that the following additional simplifications are also without loss, with proof deferred to a later draft of the paper.

- Disagreement behavior in the agent's state L is identical to agreement behavior in state L.
- Under both agreement and disagreement in state L, the agent's expected utility (in average terms) is equal to $\overline{u}_A$.
- Under disagreement in state H, the agent chooses revenue $y = -b(e)$.
- In each state, under both agreement and disagreement, the agent's continuation reward function—which in principle determines her continuation utility as a function of both her effort and the revenue she selects—is measurable with respect to her effort.

With these simplifications, it remains to identify only the agent's "good" effort schedule $e^{\mathrm{G}} : \Theta \to \mathbb{R}_+$ (under agreement in either state, and under disagreement in state L), her "bad" effort schedule $e^{\mathrm{B}} : \Theta \to \mathbb{R}_+$ (under disagreement in state H), and the continuation reward functions $w^{\mathrm{G}} : \mathbb{R}_+ \to \mathbb{R}$ and $w^{\mathrm{B}} : \mathbb{R}_+ \to \mathbb{R}$ that enforce them. The effort schedules e^{G} and e^{B} generate (good) *surplus* and *bad surplus*, respectively:

$$S(e^{\mathrm{G}}) \equiv \int_{\Theta} \big(g(e^{\mathrm{G}}(\theta)) - c(e^{\mathrm{G}}(\theta), \theta)\big)\phi(\theta)\, d\theta, \tag{1}$$

$$S(e^{\mathrm{B}}) \equiv \int_{\Theta} \big(-b(e^{\mathrm{B}}(\theta)) - c(e^{\mathrm{B}}(\theta), \theta)\big)\phi(\theta)\, d\theta, \tag{2}$$

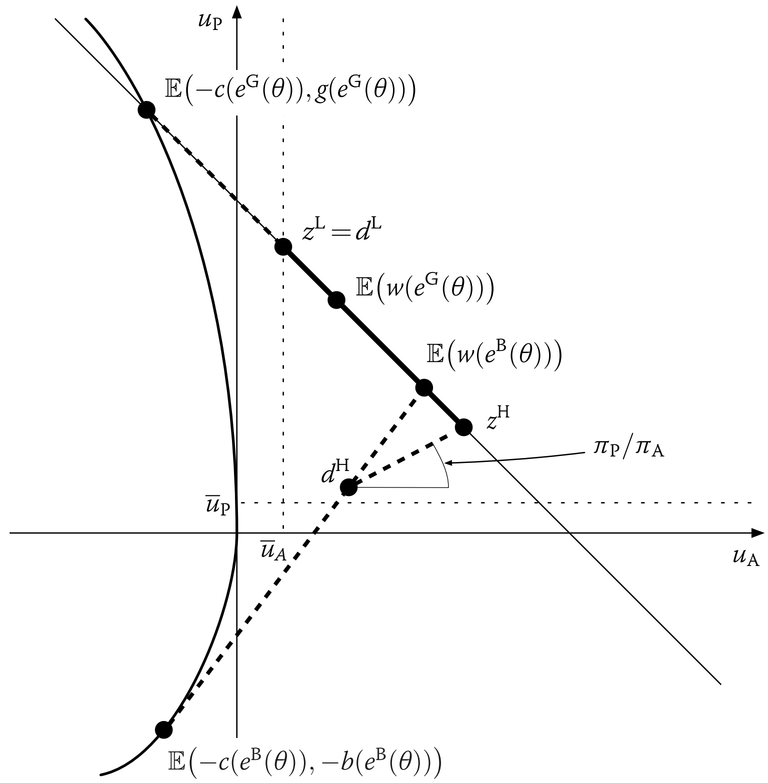

FIGURE 1

We can now establish the geometry of the contractual equilibrium in utility space, as shown in Figure 1. In Figure 1, vectors z^H and z^L are the endpoints of the *contractual equilibrium set* that reward and punish the agent, respectively. The payoff vector z^H is attained when the parties negotiate in state H, knowing that if they fail to agree then the agent will work to her bad effort schedule e^B, and then in the next period she will get a continuation reward of $w^B(\theta)$. Therefore the disagreement payoff vector in state H is

$$d^H = (1-\delta)\mathbb{E}\big(-c(e^B(\theta)), -b(e^B(\theta))\big) + \delta\mathbb{E}\big(w^B(\theta), S(e^G) - w^B(\theta)\big), \tag{3}$$

which implies that the agent's payoff under agreement is

$$\begin{aligned} z_A^H &= d_A^H + \pi_A(S(e^G) - d_A^H - d_P^H) \\ &= (1-\delta)\big(-\mathbb{E}(c(e^B(\theta))) + \pi_A(S(e^G) - S(e^B))\big) + \delta\mathbb{E}\big(w^B(\theta)\big) \end{aligned} \tag{4}$$

(where the second equality arises from substituting in Eq. (3)). Observe that d^H must weakly Pareto dominate $(\overline{u}_A, \overline{u}_P)$; otherwise one party or the other would prefer to terminate the relationship if they ever disagreed in state H. Since this constraint may not bind, it is shown as slack in Figure 1.

When the parties negotiate in state L, they understand that if they fail to agree then the agent will work to her good effort schedule e^{G}, and she will get a continuation reward of $w^{\mathrm{G}}(\theta)$ that is calibrated to deliver her expected utility of exactly $\overline{u}_A$. Therefore their disagreement payoff vector in state L is

$$d^L = z^L = (1-\delta)\mathbb{E}\big(-c(e^{\mathrm{G}}(\theta)), g(e^{\mathrm{G}}(\theta))\big) + \delta\mathbb{E}\big(w^{\mathrm{G}}(\theta), S(e^{\mathrm{G}}) - w^{\mathrm{G}}(\theta)\big). \tag{5}$$

A contractual equilibrium is identified with the solution to:

Problem 1.

$$\max_{e^{\mathrm{G}}, e^{\mathrm{B}}, w^{\mathrm{G}}, w^{\mathrm{B}}} S(e^{\mathrm{G}}) \tag{6}$$

subject to $e^{\mathrm{G}}(\theta) \geq 0$ *and* $e^{\mathrm{B}}(\theta) \geq 0$ *for all* θ*, incentive compatibility,*

$$\textit{ICG:} \quad e^{\mathrm{G}}(\theta) \in \arg\max_e -(1-\delta)\, c(e,\theta) + \delta w^{\mathrm{G}}(e) \quad \textit{for all } \theta, \tag{7}$$

$$\textit{ICB:} \quad e^{\mathrm{B}}(\theta) \in \arg\max_e -(1-\delta)\, c(e,\theta) + \delta w^{\mathrm{B}}(e) \quad \textit{for all } \theta; \tag{8}$$

individual rationality under disagreement in state H,

$$\textit{IRAH}^*\textit{:} \quad d_A^H \geq \overline{u}_A, \tag{9}$$

$$\textit{IRPH}^*\textit{:} \quad d_P^H \geq \overline{u}_P; \tag{10}$$

individual rationality under disagreement in state L (note that individual rationality in either state under disagreement implies individual rationality in that state under agreement),

$$\textit{IRAL}^*\textit{:} \quad d_A^L \geq \overline{u}_A, \tag{11}$$

$$\textit{IRPL}^*\textit{:} \quad d_P^L \geq \overline{u}_P; \tag{12}$$

and bargaining self generation,

$$\text{BSGG:} \quad w^{\mathrm{G}}(e) \in \left[z_A^L, z_A^H\right] \quad \text{for all } e, \tag{13}$$

$$\text{BSGB:} \quad w^{\mathrm{B}}(e) \in \left[z_A^L, z_A^H\right] \quad \text{for all } e. \tag{14}$$

Our main result is that under general conditions an optimal relational contract features a fully pooling bad effort schedule, and a fully pooling or partially separating good effort schedule.

Theorem 1 (Mostly proven)**.** *Under Assumptions 1 to 3, there exists a solution to Problem 1 such that* $e^{\mathrm{B}}(\theta) = \hat{e}^{\mathrm{B}}$ *for all* $\theta \in \Theta$*, where* $\hat{e}^{\mathrm{B}} \geq 0$*. For an open set of parameters under these assumptions, also* $\hat{e}^{\mathrm{B}} > 0$ *and* $e^{\mathrm{G}}(\theta) = \max\{e^{\mathrm{RG}}(\theta), \hat{e}^{\mathrm{G}}\}$*, where* e^{RG} *is strictly decreasing,* $\hat{e}^{\mathrm{G}} > 0$*, and* $\max\{\theta : e^{\mathrm{G}}(\theta) = \hat{e}^{\mathrm{G}}\} > \underline{\theta}$*.*

The rest of this draft mostly proves this theorem, though there are a few loose ends to be tied up. Based on Fig. 1, we also claim that welfare is increasing in π_A, and that z_P^H is quasiconcave in π_A, with a strictly interior maximum. Proof of these claims awaits a later draft.

4 Analysis

4.1 Simplification

Now we convert Problem 1 to a simpler problem. First, each IC constraint is satisfied if and only if appropriate envelope and monotonicity conditions are satisfied. In addition, increasing the "span" $z_A^H - z_A^L$ of continuation values allows the agent to be given higher powered incentives, so among optimal equilibria there will always exist one that maximizes this span subject to the other constraints. It follows that without loss of generality, w^{B} and w^{G} can be optimally selected to set

$$\max_e w^{\mathrm{B}}(e) = w^{\mathrm{B}}(e^{\mathrm{B}}(\underline{\theta})) = z_A^H, \tag{15}$$

$$\min_e w^{\mathrm{G}}(e) = w^{\mathrm{G}}(0) = z_A^L = d_A^L = \overline{u}_A, \tag{16}$$

$$\delta w^{\mathrm{G}}(e^{\mathrm{G}}(\underline{\theta})) - (1-\delta)c(e^{\mathrm{G}}(\underline{\theta}), \underline{\theta}) = \delta w^{\mathrm{G}}(0); \tag{17}$$

i.e., BSGB binds at the top, BSGG binds at the bottom, IRAL binds, and ICG binds for type $\underline{\theta}$ contemplating a deviation to $e = 0$. Moreover, we have already imposed $w^{\mathrm{G}}(0) = \overline{u}_A$, which implies binding IRAL. Finally, observe that IRPL is implied by IRPH. Therefore we have simplified Problem 1 to:

Problem 2.

$$\max_{e^{\mathrm{G}}, e^{\mathrm{B}}, w^{\mathrm{G}}, w^{\mathrm{B}}} S(e^{\mathrm{G}}) \tag{18}$$

subject to the envelope and monotonicity versions of incentive compatibility

$$\text{ECG:} \quad \frac{\delta}{1-\delta} w^{\mathrm{G}}(e^{\mathrm{G}}(\theta)) = \frac{\delta}{1-\delta} \overline{u}_A + c(e^{\mathrm{G}}(\theta), \theta) + \int_{\theta}^{\overline{\theta}} c_\theta(e^{\mathrm{G}}(s), s)\, ds, \tag{19}$$

$$\text{MG:} \quad e^{\mathrm{G}}(\theta) \text{ is nonincreasing and } e^{\mathrm{G}}(\overline{\theta}) \geq 0, \tag{20}$$

$$\text{ECB:} \quad \frac{\delta}{1-\delta} w^{\mathrm{B}}(e^{\mathrm{B}}(\theta)) = \frac{\delta}{1-\delta} z_A^H + c(e^{\mathrm{B}}(\theta), \theta) - c(e^{\mathrm{B}}(\underline{\theta}), \underline{\theta}) - \int_{\underline{\theta}}^{\theta} c_\theta(e^{\mathrm{B}}(s), s)\, ds, \tag{21}$$

$$\text{ICB}(0)\text{:} \quad \frac{\delta}{1-\delta} w^{\mathrm{B}}(e^{\mathrm{B}}(\overline{\theta})) \geq \frac{\delta}{1-\delta} w^{\mathrm{B}}(0) + c(e^{\mathrm{B}}(\overline{\theta}), \overline{\theta}), \tag{22}$$

$$\text{MB:} \quad e^{\mathrm{B}}(\theta) \text{ is nonincreasing and } e^{\mathrm{B}}(\overline{\theta}) \geq 0; \tag{23}$$

individual rationality in state H

$$\text{IRAH}^{**}\text{:} \quad \frac{\delta}{1-\delta} \mathbb{E} w^{\mathrm{B}}(e^{\mathrm{B}}(\theta)) \geq \frac{1}{1-\delta} \overline{u}_A + \mathbb{E} c(e^{\mathrm{B}}(\theta)), \tag{24}$$

$$\text{IRPH}^{**}\text{:} \quad \frac{\delta}{1-\delta} \big(S(e^{\mathrm{G}}) - \mathbb{E} w^{\mathrm{B}}(e^{\mathrm{B}}(\theta)) \big) \geq \frac{1}{1-\delta} \overline{u}_P + \mathbb{E} b(e^{\mathrm{B}}(\theta)); \tag{25}$$

and simplified self-generation constraints

$$\text{SG:} \quad w^{\mathrm{G}}(e^{\mathrm{G}}(\underline{\theta})) \leq z_A^H, \qquad w^{\mathrm{G}}(0) = \overline{u}_A, \tag{26}$$

$$\text{SB:} \quad w^{\mathrm{B}}(e^{\mathrm{B}}(\underline{\theta})) = z_A^H, \qquad w^{\mathrm{B}}(0) \geq \overline{u}_A. \tag{27}$$

Next we take steps to eliminate w^{B} and w^{G} from the optimization problem. Evaluating ECG at $\underline{\theta}$ and combining with SG yields the equivalent "dynamic enforcement" constraint (Levin 2003) ICDEG, below; similarly evaluating ECB at $\overline{\theta}$ and combining with SB and ICB(0) yields the dynamic enforcement constraint ICDEB. Finally, it is now evident that the maximum surplus is attained if the maximum span is attained, so it suffices to maximize z_A^H rather than $S(e^{\mathrm{G}})$. We rewrite z_A^H as:

$$
\begin{aligned}
z_A^H(e^{\mathrm{G}}, e^{\mathrm{B}}) &= (1-\delta)\big(-\mathbb{E}(c(e^{\mathrm{B}}(\theta))) + \pi_A(S(e^{\mathrm{G}}) - S(e^{\mathrm{B}}))\big) \\
&\delta z_A^H(e^{\mathrm{G}}, e^{\mathrm{B}}) + (1-\delta)\left(\mathbb{E}(c(e^{\mathrm{B}}(\theta))) - c(e^{\mathrm{B}}(\underline{\theta}), \underline{\theta}) - \mathbb{E}\int_{\underline{\theta}}^{\theta} c_\theta(e^{\mathrm{B}}(s), s)\, ds\right) \\
&= \pi_A(S(e^{\mathrm{G}}) - S(e^{\mathrm{B}})) - c(e^{\mathrm{B}}(\underline{\theta}), \underline{\theta}) - \mathbb{E}\int_{\underline{\theta}}^{\theta} c_\theta(e^{\mathrm{B}}(s), s)\, ds,
\end{aligned}
\tag{28}
$$

where the first equality arises from taking expectations of ECB and substituting into Eq. (4)). Now we have a consolidated and simplified "Main" Problem that is equivalent to Problems 1 and 2.

Problem 3 (Main Problem)**.**

$$
\max_{e^{\mathrm{G}}, e^{\mathrm{B}}} z_A^H(e^{\mathrm{G}}, e^{\mathrm{B}}) \tag{29}
$$

subject to the dynamic enforcement constraints

$$
\text{ICDEG:} \quad \frac{\delta}{1-\delta}\big(z_A^H(e^{\mathrm{G}}, e^{\mathrm{B}}) - \overline{u}_A\big) \geq c(e^{\mathrm{G}}(\underline{\theta}), \underline{\theta}) + \int_{\underline{\theta}}^{\overline{\theta}} c_\theta(e^{\mathrm{G}}(s), s)\, ds, \tag{30}
$$

$$
\text{ICDEB:} \quad \frac{\delta}{1-\delta}\big(z_A^H(e^{\mathrm{G}}, e^{\mathrm{B}}) - \overline{u}_A\big) \geq c(e^{\mathrm{B}}(\underline{\theta}), \underline{\theta}) + \int_{\underline{\theta}}^{\overline{\theta}} c_\theta(e^{\mathrm{B}}(s), s)\, ds; \tag{31}
$$

the monotonicity constraints

$$
\text{MG:} \quad e^{\mathrm{G}}(\theta) \text{ is nonincreasing and } e^{\mathrm{G}}(\overline{\theta}) \geq 0, \tag{32}
$$

$$
\text{MB:} \quad e^{\mathrm{B}}(\theta) \text{ is nonincreasing and } e^{\mathrm{B}}(\overline{\theta}) \geq 0; \tag{33}
$$

and individual rationality constraints

$$
\text{IRAH:} \quad \frac{\delta}{1-\delta}\big(z_A^H(e^{\mathrm{G}}, e^{\mathrm{B}}) - \overline{u}_A\big) \geq c(e^{\mathrm{B}}(\underline{\theta}), \underline{\theta}) + \mathbb{E}\int_{\underline{\theta}}^{\theta} c_\theta(e^{\mathrm{B}}(s), s)\, ds + \overline{u}_A, \tag{34}
$$

$$
\begin{aligned}
\text{IRPH:} \quad & \frac{\delta}{1-\delta} z_A^H + \mathbb{E}c(e^{\mathrm{B}}(\theta), \theta) - c(e^{\mathrm{B}}(\underline{\theta}), \underline{\theta}) - \mathbb{E}\int_{\underline{\theta}}^{\theta} c_\theta(e^{\mathrm{B}}(s), s)\, ds \\
&\leq \frac{\delta}{1-\delta} S(e^{\mathrm{G}}) - \frac{1}{1-\delta}\overline{u}_P - \mathbb{E}b(e^{\mathrm{B}}(\theta)).
\end{aligned}
\tag{35}
$$

Because the problem is convex, in any optimal contract each effort schedule (e^{B} and e^{G}) is optimal when the other is held fixed. Since the objective and all the constraints are additively separable in the two effort schedules, this enables us to break the optimization problem into two smaller problems, one for each effort schedule.

Problem 4 ("Good" Problem)**.**

$$\max_{e^{\mathrm{G}}} z_A^H(e^{\mathrm{G}}, \overline{e^{\mathrm{B}}}) \tag{36}$$

for some fixed $\overline{e^{\mathrm{B}}}$*, subject to*

$$\text{ICDEG:} \quad \frac{\delta}{1-\delta}\left(z_A^H(e^{\mathrm{G}}, \overline{e^{\mathrm{B}}}) - \overline{u}_A\right) \geq c(e^{\mathrm{G}}(\underline{\theta}), \underline{\theta}) + \int_{\underline{\theta}}^{\overline{\theta}} c_\theta(e^{\mathrm{G}}(\theta), \theta)\, d\theta, \tag{37}$$

$$\text{MG:} \quad e^{\mathrm{G}}(\theta) \text{ is nonincreasing and } e^{\mathrm{G}}(\overline{\theta}) \geq 0. \tag{38}$$

Problem 5 ("Bad" Problem)**.**

$$\max_{e^{\mathrm{B}}} z_A^H(\overline{e^{\mathrm{G}}}, e^{\mathrm{B}}) \tag{39}$$

for some fixed $\overline{e^{\mathrm{G}}}$*, subject to*

$$\text{ICDEB:} \quad \frac{\delta}{1-\delta}\left(z_A^H(\overline{e^{\mathrm{G}}}, e^{\mathrm{B}}) - \overline{u}_A\right) \geq c(e^{\mathrm{B}}(\underline{\theta}), \underline{\theta}) + \int_{\underline{\theta}}^{\overline{\theta}} c_\theta(e^{\mathrm{B}}(s), s)\, ds; \tag{40}$$

$$\text{MB:} \quad e^{\mathrm{B}}(\theta) \text{ is nonincreasing and } e^{\mathrm{B}}(\overline{\theta}) \geq 0; \tag{41}$$

$$\text{IRAH:} \quad \frac{\delta}{1-\delta}\left(z_A^H(\overline{e^{\mathrm{G}}}, e^{\mathrm{B}}) - \overline{u}_A\right) \geq c(e^{\mathrm{B}}(\underline{\theta}), \underline{\theta}) + \mathbb{E}\int_{\underline{\theta}}^{\theta} c_\theta(e^{\mathrm{B}}(s), s)\, ds + \overline{u}_A, \tag{42}$$

$$\text{IRPH:} \quad \begin{aligned} &\frac{\delta}{1-\delta} z_A^H + \mathbb{E}c(e^{\mathrm{B}}(\theta), \theta) - c(e^{\mathrm{B}}(\underline{\theta}), \underline{\theta}) - \mathbb{E}\int_{\underline{\theta}}^{\theta} c_\theta(e^{\mathrm{B}}(s), s)\, ds \\ &\qquad \leq \frac{\delta}{1-\delta} S(\overline{e^{\mathrm{G}}}) - \frac{1}{1-\delta}\overline{u}_P - \mathbb{E}b(e^{\mathrm{B}}(\theta)). \end{aligned} \tag{43}$$

Let $e^{\mathrm{G}*}\big(\overline{e^{\mathrm{B}}}\big)$ be the solution to Problem 4, and $e^{\mathrm{B}*}\big(\overline{e^{\mathrm{G}}}\big)$ be the solution to Problem 5. Then we find the solution to Problem 1 at a fixed point $(e^{\mathrm{G}}, e^{\mathrm{B}}) = \big(e^{\mathrm{G}*}(e^{\mathrm{B}}), e^{\mathrm{B}*}(e^{\mathrm{G}})\big)$.

4.2 Solution to the Good Problem

Following Levin (2003), we express Problem 4 as an optimal control problem in θ, with $e^{\mathrm{G}}(\theta)$ as a state and $\gamma(\theta) = \dot{e}^{\mathrm{G}}(\theta)$ as the control. We are interested only in cases in which the solution is for the agent to put in non-zero effort, so we ignore the non-negativity constraint on effort during optimization, and then simply note that the constraint is satisfied at the solution. Since the ICDEG constraint involves a definite integral over the state space, we introduce an auxiliary state $K(\theta)$, such that ICDEG can be expressed in terms of $K(\overline{\theta})$. Now we substitute for $z_A^H(e^{\mathrm{G}}, \overline{e^{\mathrm{B}}})$ to transform Problem 4 into an optimal control problem in standard form, for a given bad effort schedule $\overline{e^{\mathrm{B}}}$.

Problem 6 (Good Problem in Standard Form)**.**

$$\max_{e^{\mathrm{G}}(\theta)} \left(\begin{aligned} &\pi_A \int_{\underline{\theta}}^{\overline{\theta}} \left(g(e^{\mathrm{G}}(\theta)) - c(e^{\mathrm{G}}(\theta), \theta) \right) \phi(\theta)\, d\theta \\ &\quad - \pi_A S(\overline{e^{\mathrm{B}}}) - c(\overline{e^{\mathrm{B}}}(\underline{\theta}), \underline{\theta}) - \int_{\underline{\theta}}^{\overline{\theta}} \phi(\theta) \int_{\underline{\theta}}^{\theta} c_\theta(\overline{e^{\mathrm{B}}}(s), s)\, ds\, d\theta \end{aligned} \right) \tag{44}$$

subject to

$$\dot{e^{\mathrm{G}}}(\theta) = \gamma^{\mathrm{G}}(\theta), \tag{45}$$

$$\dot{K^{\mathrm{G}}}(\theta) = \frac{\delta}{1-\delta} \pi_A \left(g(e^{\mathrm{G}}(\theta)) - c(e^{\mathrm{G}}(\theta), \theta) \right) \phi(\theta) - c_\theta(e^{\mathrm{G}}(\theta), \theta), \tag{46}$$

$$-\gamma^{\mathrm{G}}(\theta) \geq 0, \tag{47}$$

$$K^{\mathrm{G}}(\underline{\theta}) = 0, \quad \underline{\theta} = \theta_L, \quad \overline{\theta} = \theta_H, \tag{48}$$

$$\begin{aligned} &K^{\mathrm{G}}(\overline{\theta}) - c(e^{\mathrm{G}}(\underline{\theta}), \underline{\theta}) - \frac{\delta}{1-\delta} u_A \\ &\quad - \frac{\delta}{1-\delta} \left(\pi_A S(\overline{e^{\mathrm{B}}}) + c(\overline{e^{\mathrm{B}}}(\underline{\theta}), \underline{\theta}) + \int_{\underline{\theta}}^{\overline{\theta}} \phi(\theta) \int_{\underline{\theta}}^{\theta} c_\theta(\overline{e^{\mathrm{B}}}(s), s)\, ds\, d\theta \right) \\ &\quad \geq 0, \end{aligned} \tag{49}$$

with co-state variables $\eta^{\mathrm{G}}(\theta)$, and $\lambda(\theta)$ for the states $e^{\mathrm{G}}(\theta)$ and $K^{\mathrm{G}}(\theta)$ respectively, and multipliers $\nu^{\mathrm{G}}(\theta)$, $(\mu_1^{\mathrm{G}}, \mu_2^{\mathrm{G}}, \mu_3^{\mathrm{G}})$, and μ^{G} for the equality and inequality constraints Eqs. (47) to (49), respectively.

The good problem is quite similar to the optimization problem studied by Levin (2003), differing mainly in the presence of the coefficient π_A and the constant terms involving $\overline{e^{\mathrm{B}}}$. Accordingly, we reach similar conclusions, proven in Appendix A: If ICDEG does not bind, fully-separating first best is attainable; if ICDEG does bind, then there may be either partial pooling (with separation for high types) or full pooling. Our analysis will hinge on μ^{G}, the multiplier for the ICDEG constraint.

If δ and π_A are sufficiently large, u_A is sufficiently small, and $\overline{e^{\mathrm{B}}}$ is such that the term in parentheses on the second line of Eq. (49) is sufficiently large (recall that $S(\overline{e^{\mathrm{B}}})$ is negative), then ICDEG will be slack. If so, it suffices to maximize $z_A^H(e^{\mathrm{G}}, \overline{e^{\mathrm{B}}})$ subject to the monotonicity constraint MG. Because first-best effort e^{FB} is decreasing, it both maximizes z_A^H and satisfies monotonicity.

If instead ICDEG binds, there can either be full pooling or partial pooling. First consider full pooling. In this case matters are simple. Let $e^{\mathrm{G}}(\theta) = \hat{e}^{\mathrm{G}}$ for all θ. Simply solve for $\hat{e}^{\mathrm{G}}$ from the binding ICDEG constraint.

The other possibility is partial pooling. In this case there exists a cutoff type $\hat{\theta} \in (\underline{\theta}, \overline{\theta})$ and strictly decreasing function $e^{\mathrm{RG}} : \Theta \to \mathbb{R}_+$ such that high-cost types $\theta > \hat{\theta}$ separate by exerting effort $e^{\mathrm{RG}}(\theta)$ while low-cost types $\theta \leq \hat{\theta}$ exert pooling effort $e^{\mathrm{RG}}(\hat{\theta})$. In particular, e^{RG} is the unique solution to

$$\big(g_e(e(\theta)) - c_e(e(\theta), \theta)\big)\phi(\theta) = \frac{\mu^{\mathrm{G}} c_{e\theta}(e(\theta), \theta)}{\pi_A\big(1 + \mu^{\mathrm{G}} \frac{\delta}{1-\delta}\big)}. \tag{50}$$

Under our assumptions, e^{RG} is strictly decreasing, and $e^{\mathrm{RG}}(\theta) < e^{FB}(\theta)$ for all θ. To find the cutoff type $\hat{\theta}$, observe that $\nu^{\mathrm{G}}(\underline{\theta}) = \mu^{\mathrm{G}} c_e(e^{\mathrm{G}}(\underline{\theta}), \underline{\theta})$ and $\nu^{\mathrm{G}}(\hat{\theta}) = 0$. If we integrate Eq. (169) from $\underline{\theta}$ to $\hat{\theta}$ and apply the boundary conditions, we have:

$$\int_{\underline{\theta}}^{\hat{\theta}} \big(g_e(e^{\mathrm{G}}(\theta)) - c_e(e^{\mathrm{G}}(\theta), \theta)\big)\, \phi(\theta)\, d\theta = \frac{\mu^{\mathrm{G}} c_e(e^{\mathrm{G}}(\hat{\theta}), \hat{\theta})}{\pi_A\left(1 + \mu^{\mathrm{G}} \frac{\delta}{1-\delta}\right)}. \tag{51}$$

From the assumptions in the primitives, there is at most one $\hat{\theta} \in (\underline{\theta}, \overline{\theta})$ that satisfies this condition.

4.3 Solution to the Bad Problem

With the extra constraints (IRAH and IRPH), Problem 5 is a more complex optimal control problem than Problem 4. We first substitute for z_A^H, which Problem 5 maximizes, as follows:[3]

$$\begin{aligned} z_A^H(\overline{e^{\mathrm{G}}}, e^{\mathrm{B}}) &= \pi_A\big(S(\overline{e^{\mathrm{G}}}) - S(e^{\mathrm{B}})\big) - c(e^{\mathrm{B}}(\underline{\theta}), \underline{\theta}) - \int_{\underline{\theta}}^{\overline{\theta}} \phi(\theta) \int_{\underline{\theta}}^{\theta} c_\theta(e^{\mathrm{B}}(s), s)\, ds\, d\theta \\ &= \int_{\underline{\theta}}^{\overline{\theta}} \left(\pi_A \left(b(e^{\mathrm{B}}(\theta)) + c(e^{\mathrm{B}}(\theta), \theta)\right) \phi(\theta) - c_\theta(e^{\mathrm{B}}(\theta), \theta)(1 - \Phi(\theta))\right) d\theta \\ &\quad - c(e^{\mathrm{B}}(\underline{\theta}), \underline{\theta}) + \pi_A S(\overline{e^{\mathrm{G}}}) \end{aligned} \tag{52}$$

[3] Observe that $\int_{\underline{\theta}}^{\overline{\theta}} \phi(\theta) \int_{\underline{\theta}}^{\theta} c_\theta(e^{\mathrm{B}}(s), s)\, ds\, d\theta = \int_{\underline{\theta}}^{\overline{\theta}} c_\theta(e^{\mathrm{B}}(\theta), \theta) \int_{\theta}^{\overline{\theta}} \phi(s)\, ds\, d\theta$.

With the objective in this form, it is evident that if there were no constraints then the optimal counter-productive effort of the most costly types (for whom $1 - \Phi(\theta) = 0$) would be infinite. Therefore even if ICDEB, IRAH, and IRPH do not bind, it must be that MB binds. That is, in the bad problem there is no analog of the strictly decreasing first best solution to the good problem.

4.3.1 Constructing the additional state variables

To express the constraints in appropriate form for analysis, we define three state variables in addition to $e^{\mathrm{B}}(\theta)$: $K^{\mathrm{B}}(\theta)$ for the ICDEB constraint, $L(\theta)$ for the IRAH constraint, and $M(\theta)$ for the IRPH constraint. We rewrite ICDEB as

$$\begin{aligned}&\frac{\delta}{1-\delta}\left(z_A^H(\overline{e^{\mathrm{G}}}, e^{\mathrm{B}}) - \overline{u}_A\right) \geq c(e^{\mathrm{B}}(\underline{\theta}), \underline{\theta}) + \int_{\underline{\theta}}^{\overline{\theta}} c_\theta(e^{\mathrm{B}}(\theta), \theta)\, d\theta \\ \iff & K^{\mathrm{B}}(\overline{\theta}) + \frac{\delta}{1-\delta}\left(\pi_A S(\overline{e^{\mathrm{G}}}) - \overline{u}_A\right) - \frac{1}{1-\delta} c(e^{\mathrm{B}}(\underline{\theta}), \underline{\theta}) \geq 0,\end{aligned} \tag{53}$$

where

$$K^{\mathrm{B}}(\theta) \equiv \int_{\underline{\theta}}^{\theta} \left(\frac{\delta}{1-\delta} \begin{pmatrix} \pi_A \left(b(e^{\mathrm{B}}(\tilde{\theta})) + c(e^{\mathrm{B}}(\tilde{\theta}), \tilde{\theta})\right) \phi(\tilde{\theta}) \\ - c_\theta(e^{\mathrm{B}}(\tilde{\theta}), \tilde{\theta})(1 - \Phi(\tilde{\theta})) \end{pmatrix} - c_\theta(e^{\mathrm{B}}(\tilde{\theta}), \tilde{\theta}) \right) d\tilde{\theta}. \tag{54}$$

We rewrite IRAH as

$$\begin{aligned}&\frac{\delta}{1-\delta}\left(z_A^H(\overline{e^{\mathrm{G}}}, e^{\mathrm{B}}) - \overline{u}_A\right) \geq c(e^{\mathrm{B}}(\underline{\theta}), \underline{\theta}) + \int_{\underline{\theta}}^{\overline{\theta}} c_\theta(e^{\mathrm{B}}(\theta), \theta)(1 - \Phi(\theta))\, d\theta + \overline{u}_A \\ \iff & L(\overline{\theta}) + \frac{\delta}{1-\delta}\pi_A S(\overline{e^{\mathrm{G}}}) - \frac{1}{1-\delta}\left(c(e^{\mathrm{B}}(\underline{\theta}), \underline{\theta}) + \overline{u}_A\right) \geq 0\end{aligned} \tag{55}$$

where

$$L(\theta) \equiv \frac{1}{1-\delta} \int_{\underline{\theta}}^{\theta} \left(\delta \pi_A \left(b(e^{\mathrm{B}}(\tilde{\theta})) + c(e^{\mathrm{B}}(\tilde{\theta}), \tilde{\theta}) \right) \phi(\tilde{\theta}) - c_\theta(e^{\mathrm{B}}(\tilde{\theta}), \tilde{\theta})(1 - \Phi(\tilde{\theta})) \right) d\tilde{\theta}. \tag{56}$$

We rewrite IRPH as

$$\begin{aligned}
&\frac{\delta}{1-\delta} z_A^H(\overline{e^{\mathrm{G}}}, e^{\mathrm{B}}) + \int_{\underline{\theta}}^{\overline{\theta}} c(e^{\mathrm{B}}(\theta), \theta)\phi(\theta)\, d\theta - c(e^{\mathrm{B}}(\underline{\theta}), \underline{\theta}) - \int_{\underline{\theta}}^{\overline{\theta}} c_\theta(e^{\mathrm{B}}(\theta), \theta)(1-\Phi(\theta))\, d\theta \\
&\quad \leq \frac{\delta}{1-\delta} S(\overline{e^{\mathrm{G}}}) - \frac{1}{1-\delta}\overline{u}_P - \int_{\underline{\theta}}^{\overline{\theta}} b(e^{\mathrm{B}}(\theta))\phi(\theta)\, d\theta \\
&\iff -M(\overline{\theta}) + \frac{\delta}{1-\delta}(1-\pi_A) S(\overline{e^{\mathrm{G}}}) + \frac{1}{1-\delta}\left(c(e^{\mathrm{B}}(\underline{\theta}), \underline{\theta}) - \overline{u}_P\right) \geq 0,
\end{aligned} \tag{57}$$

where

$$M(\theta) \equiv \int_{\underline{\theta}}^{\theta} \begin{pmatrix} \left(\frac{\delta \pi_A}{1-\delta} + 1\right)\left(b(e^{\mathrm{B}}(\tilde{\theta})) + c(e^{\mathrm{B}}(\tilde{\theta}), \tilde{\theta})\right)\phi(\tilde{\theta}) \\ -\frac{1}{1-\delta} c_\theta(e^{\mathrm{B}}(\tilde{\theta}), \tilde{\theta})(1-\Phi(\tilde{\theta})) \end{pmatrix} d\tilde{\theta}. \tag{58}$$

4.3.2 Characterization

Now we can express Problem 5 in standard form (Léonard and van Long 1992, Problem 7.114–7.115), and derive the necessary conditions for an optimum.

Problem 7 (Bad Problem in Standard Form)**.**

$$\max_{e^{\mathrm{B}}} \begin{pmatrix} \int_{\underline{\theta}}^{\overline{\theta}} \left(\pi_A \left(b(e^{\mathrm{B}}(\theta)) + c(e^{\mathrm{B}}(\theta), \theta)\right)\phi(\theta) - c_\theta(e^{\mathrm{B}}(\theta), \theta)(1-\Phi(\theta))\right) d\theta \\ - c(e^{\mathrm{B}}(\underline{\theta}), \underline{\theta}) + \pi_A S(\overline{e^{\mathrm{G}}}) \end{pmatrix} \tag{59}$$

subject to

$$\dot{e^{\mathrm{B}}}(\theta) = \gamma^{\mathrm{B}}(\theta) \tag{60}$$

$$\dot{K}^{\mathrm{B}}(\theta) = \frac{\delta}{1-\delta}\left(\pi_A\left(b(e^{\mathrm{B}}(\theta)) + c(e^{\mathrm{B}}(\theta),\theta)\right)\phi(\theta) - c_\theta(e^{\mathrm{B}}(\theta),\theta)(1-\Phi(\theta))\right) - c_\theta(e^{\mathrm{B}}(\theta),\theta) \tag{61}$$

$$\dot{L}(\theta) = \frac{1}{1-\delta}\left(\delta\pi_A\left(b(e^{\mathrm{B}}(\theta)) + c(e^{\mathrm{B}}(\theta),\theta)\right)\phi(\theta) - c_\theta(e^{\mathrm{B}}(\theta),\theta)(1-\Phi(\theta))\right) \tag{62}$$

$$\dot{M}(\theta) = \left(\frac{\delta\pi_A}{1-\delta}+1\right)\left(b(e^{\mathrm{B}}(\theta)) + c(e^{\mathrm{B}}(\theta),\theta)\right)\phi(\theta) - \frac{1}{1-\delta}c_\theta(e^{\mathrm{B}}(\theta),\theta)(1-\Phi(\theta)) \tag{63}$$

$$-\gamma^{\mathrm{B}}(\theta) \geq 0 \tag{64}$$

$$e^{\mathrm{B}}(\overline{\theta}) \geq 0 \tag{65}$$

$$K^{\mathrm{B}}(\underline{\theta}) = 0,\ L(\underline{\theta}) = 0,\ M(\underline{\theta}) = 0,\ \underline{\theta} = \theta_L, \text{ and } \overline{\theta} = \theta_H \tag{66}$$

$$K^{\mathrm{B}}(\overline{\theta}) + \frac{\delta}{1-\delta}\left(\pi_A S(\overline{e^{\mathrm{G}}}) - \overline{u}_A\right) - \frac{1}{1-\delta}c(e^{\mathrm{B}}(\underline{\theta}),\underline{\theta}) \geq 0 \tag{67}$$

$$L(\overline{\theta}) + \frac{\delta}{1-\delta}\pi_A S(\overline{e^{\mathrm{G}}}) - \frac{1}{1-\delta}\left(c(e^{\mathrm{B}}(\underline{\theta}),\underline{\theta}) + \overline{u}_A\right) \geq 0 \tag{68}$$

$$-M(\overline{\theta}) + \frac{\delta(1-\pi_A)S(\overline{e^{\mathrm{G}}})}{1-\delta} + \frac{1}{1-\delta}\left(c(e^{\mathrm{B}}(\underline{\theta}),\underline{\theta}) - \overline{u}_P\right) \geq 0 \tag{69}$$

with co-state variables $\eta^{\mathrm{B}}(\theta)$, $\lambda_1(\theta)$, $\lambda_2(\theta)$, and $\lambda_3(\theta)$ for the states $e^{\mathrm{B}}(\theta)$, $K^{\mathrm{B}}(\theta)$, $L(\theta)$, and $M(\theta)$, respectively; and multipliers $\nu^{\mathrm{B}}(\theta)$, μ_0^{B}, $(\mu_1^{\mathrm{B}}, \mu_2^{\mathrm{B}}, \mu_3^{\mathrm{B}}, \mu_4, \mu_5)$, μ^{B}, ξ, and ζ for the equality and inequality constraints Eqs. (64) *to* (69)*, respectively.*

Necessary conditions The Hamiltonian for this problem is:

$$
\begin{aligned}
\mathcal{H}^{\mathrm{B}}(\theta) &= \pi_A \left(b(e^{\mathrm{B}}(\theta)) + c(e^{\mathrm{B}}(\theta),\theta)\right)\phi(\theta) - c_\theta(e^{\mathrm{B}}(\theta),\theta)(1-\Phi(\theta)) + \eta^{\mathrm{B}}(\theta)\gamma^{\mathrm{B}}(\theta) \\
&\quad + \lambda_1(\theta)\left(\frac{\delta}{1-\delta}\left(\pi_A\left(b(e^{\mathrm{B}}(\theta)) + c(e^{\mathrm{B}}(\theta),\theta)\right)\phi(\theta) - c_\theta(e^{\mathrm{B}}(\theta),\theta)(1-\Phi(\theta))\right) - c_\theta(e^{\mathrm{B}}(\theta),\theta)\right) \\
&\quad + \lambda_2(\theta)\left(\frac{\delta}{1-\delta}\pi_A\left(b(e^{\mathrm{B}}(\theta)) + c(e^{\mathrm{B}}(\theta),\theta)\right)\phi(\theta) - \frac{1}{1-\delta}c_\theta(e^{\mathrm{B}}(\theta),\theta)(1-\Phi(\theta))\right) \\
&\quad + \lambda_3(\theta)\left(\left(\frac{\delta\pi_A}{1-\delta}+1\right)\left(b(e^{\mathrm{B}}(\theta)) + c(e^{\mathrm{B}}(\theta),\theta)\right)\phi(\theta) - \frac{1}{1-\delta}c_\theta(e^{\mathrm{B}}(\theta),\theta)(1-\Phi(\theta))\right) \\
&= \left(\pi_A + \frac{\delta\pi_A}{1-\delta}\left(\lambda_1(\theta)+\lambda_2(\theta)\right) + \lambda_3(\theta)\left(\frac{\delta\pi_A}{1-\delta}+1\right)\right)\left(b(e^{\mathrm{B}}(\theta)) + c(e^{\mathrm{B}}(\theta),\theta))\right)\phi(\theta) \\
&\quad - \left(\left(1+\lambda_1(\theta)\frac{\delta}{1-\delta} + \frac{1}{1-\delta}(\lambda_2(\theta)+\lambda_3(\theta))\right)(1-\Phi(\theta)) + \lambda_1(\theta)\right)c_\theta(e^{\mathrm{B}}(\theta),\theta) \\
&\quad + \eta^{\mathrm{B}}(\theta)\gamma^{\mathrm{B}}(\theta),
\end{aligned}
\tag{70}
$$

and the Lagrangean is

$$
\mathcal{L}^{\mathrm{B}} = \mathcal{H}^{\mathrm{B}} - \nu^{\mathrm{B}}(\theta)\gamma^{\mathrm{B}}(\theta). \tag{71}
$$

By Theorem 6.5.1 of Léonard and van Long 1992), the necessary conditions for a solution to Problem 7 are as follows.

1. There exists a piecewise-continuous multiplier $\nu^{\mathrm{B}}(\theta)$ such that, for all θ,

$$
\frac{\partial \mathcal{L}^*}{\partial \gamma^{\mathrm{B}}(\theta)} = \eta^{\mathrm{B}}(\theta) - \nu^{\mathrm{B}}(\theta) = 0, \tag{72}
$$

$$
\nu^{\mathrm{B}}(\theta) \geq 0, \qquad -\gamma^{\mathrm{B}}(\theta) \geq 0, \qquad \nu^{\mathrm{B}}(\theta)\gamma^{\mathrm{B}}(\theta) = 0. \tag{73}
$$

2. The co-state variables $\eta^{\mathrm{B}}(\theta)$, $\lambda_1(\theta)$, $\lambda_2(\theta)$, and $\lambda_3(\theta)$ are continuous, and have piecewise-

continuous derivatives satisfying the following conditions:

$$\dot{\eta}^B(\theta) = -\frac{\partial \mathcal{L}^{\mathrm{B}}}{\partial e^{\mathrm{B}}(\theta)}$$
$$= -\left(\pi_A + \frac{\delta\pi_A}{1-\delta}\left(\lambda_1(\theta)+\lambda_2(\theta)\right) + \lambda_3(\theta)\left(\frac{\delta\pi_A}{1-\delta}+1\right)\right)\left(b_e(e^{\mathrm{B}}(\theta)) + c_e(e^{\mathrm{B}}(\theta),\theta)\right)\right)\phi(\theta)$$
$$+\left(\left(1+\lambda_1(\theta)\frac{\delta}{1-\delta} + \frac{1}{1-\delta}(\lambda_2(\theta)+\lambda_3(\theta))\right)(1-\Phi(\theta)) + \lambda_1(\theta)\right) c_{e\theta}(e^{\mathrm{B}}(\theta),\theta) \tag{74}$$

$$\dot{\lambda_1}(\theta) = -\frac{\partial \mathcal{L}^{\mathrm{B}*}}{\partial K^{\mathrm{B}}(\theta)} = -\mathcal{H}^{\mathrm{B}}_K = 0 \tag{75}$$

$$\dot{\lambda_2}(\theta) = -\frac{\partial \mathcal{L}^{\mathrm{B}*}}{\partial L(\theta)} = -\mathcal{H}^{\mathrm{B}}_L = 0 \tag{76}$$

$$\dot{\lambda_3}(\theta) = -\frac{\partial \mathcal{L}^{\mathrm{B}*}}{\partial M(\theta)} = -\mathcal{H}^{\mathrm{B}}_M = 0 \tag{77}$$

3. The state transitions satisfy

$$\dot{e}^{\mathrm{B}}(\theta) = \gamma^{\mathrm{B}}(\theta) \tag{78}$$

$$\dot{K}^{\mathrm{B}}(\theta) = \frac{\delta}{1-\delta}\left(\pi_A\left(b(e^{\mathrm{B}}(\theta)) + c(e^{\mathrm{B}}(\theta),\theta)\right)\phi(\theta) - c_\theta(e^{\mathrm{B}}(\theta),\theta)(1-\Phi(\theta))\right) - c_\theta(e^{\mathrm{B}}(\theta),\theta) \tag{79}$$

$$\dot{L}(\theta) = \frac{1}{1-\delta}\left(\delta\pi_A\left(b(e^{\mathrm{B}}(\theta)) + c(e^{\mathrm{B}}(\theta),\theta)\right)\phi(\theta) - c_\theta(e^{\mathrm{B}}(\theta),\theta)(1-\Phi(\theta))\right) \tag{80}$$

$$\dot{M}(\theta) = \left(\frac{\delta\pi_A}{1-\delta}+1\right)\left(b(e^{\mathrm{B}}(\theta)) + c(e^{\mathrm{B}}(\theta),\theta)\right)\phi(\theta) - \frac{1}{1-\delta}c_\theta(e^{\mathrm{B}}(\theta),\theta)(1-\Phi(\theta)) \tag{81}$$

4. The Lagrangean $\mathcal{L}^{\mathrm{B}}(e^{\mathrm{B}}(\theta)^*, K^{\mathrm{B}}(\theta)^*, L(\theta)^*, M(\theta)^*, \eta^{\mathrm{B}}(\theta), \lambda_1(\theta), \lambda_2(\theta), \lambda_3(\theta), \nu^{\mathrm{B}}(\theta), \theta) = \psi^{\mathrm{B}}(\theta)$ is a continuous function of θ. On each interval of continuity of $\gamma^{\mathrm{B}*}(\theta)$, $\psi^{\mathrm{B}}(\theta)$ is differentiable, and $\psi^{\mathrm{B}'}(\theta) \equiv \frac{d\mathcal{L}^{\mathrm{B}*}}{d\theta} = \frac{\partial \mathcal{L}^{\mathrm{B}*}}{\partial\theta}$.

5. The transversality conditions are satisfied:

(a)

$$\begin{aligned}
&\mathcal{H}^{\mathrm{B}}(\underline{\theta}) - \mu_1^{\mathrm{B}}\frac{\partial K^{\mathrm{B}}(\underline{\theta})}{\partial\underline{\theta}} - \mu_2^{\mathrm{B}}\frac{\partial L(\underline{\theta})}{\partial\underline{\theta}} - \mu_3^{\mathrm{B}}\frac{\partial M(\underline{\theta})}{\partial\underline{\theta}} - \mu_4\frac{\partial(\underline{\theta}-\theta_L)}{\partial\underline{\theta}} - \mu_5\frac{\partial(\overline{\theta}-\theta_H)}{\partial\underline{\theta}} \\
&\quad - \mu^{\mathrm{B}}\frac{\partial}{\partial\underline{\theta}}\left(K^{\mathrm{B}}(\overline{\theta}) + \frac{\delta}{1-\delta}\left(\pi_A S(\overline{e^{\mathrm{G}}}) - \overline{u}_A\right) - \frac{1}{1-\delta}c(e^{\mathrm{B}}(\underline{\theta}),\underline{\theta})\right) \\
&\quad - \xi\frac{\partial}{\partial\underline{\theta}}\left(L(\overline{\theta}) + \frac{\delta}{1-\delta}\pi_A S(\overline{e^{\mathrm{G}}}) - \frac{1}{1-\delta}\left(c(e^{\mathrm{B}}(\underline{\theta}),\underline{\theta}) + \overline{u}_A\right)\right) \\
&\quad - \zeta\frac{\partial}{\partial\underline{\theta}}\left(-M(\overline{\theta}) + \frac{\delta(1-\pi_A)S(\overline{e^{\mathrm{G}}})}{1-\delta} + \frac{1}{1-\delta}\left(c(e^{\mathrm{B}}(\underline{\theta}),\underline{\theta}) - \overline{u}_P\right)\right) = 0
\end{aligned} \tag{82}$$

$$\Rightarrow \quad \mathcal{H}^{\mathrm{B}}(\underline{\theta}) - \mu_4 + \frac{1}{1-\delta}c_\theta(e^{\mathrm{B}}(\underline{\theta}),\underline{\theta})(\mu^{\mathrm{B}} + \xi - \zeta) = 0 \tag{83}$$

(b)

$$\begin{aligned}
&\eta^{\mathrm{B}}(\underline{\theta}) + \mu_1^{\mathrm{B}}\frac{\partial K^{\mathrm{B}}(\underline{\theta})}{\partial e(\underline{\theta})} + \mu_2^{\mathrm{B}}\frac{\partial L(\underline{\theta})}{\partial e(\underline{\theta})} + \mu_3^{\mathrm{B}}\frac{\partial M(\underline{\theta})}{\partial e(\underline{\theta})} + \mu_4\frac{\partial(\underline{\theta}-\theta_L)}{\partial e(\underline{\theta})} + \mu_5\frac{\partial(\overline{\theta}-\theta_H)}{\partial e(\underline{\theta})} \\
&\quad + \mu^{\mathrm{B}}\frac{\partial}{\partial e(\underline{\theta})}\left(K^{\mathrm{B}}(\overline{\theta}) + \frac{\delta}{1-\delta}\left(\pi_A S(\overline{e^{\mathrm{G}}}) - \overline{u}_A\right) - \frac{1}{1-\delta}c(e^{\mathrm{B}}(\underline{\theta}),\underline{\theta})\right) \\
&\quad + \xi\frac{\partial}{\partial e(\underline{\theta})}\left(L(\overline{\theta}) + \frac{\delta}{1-\delta}\pi_A S(\overline{e^{\mathrm{G}}}) - \frac{1}{1-\delta}\left(c(e^{\mathrm{B}}(\underline{\theta}),\underline{\theta}) + \overline{u}_A\right)\right) \\
&\quad + \zeta\frac{\partial}{\partial e(\underline{\theta})}\left(-M(\overline{\theta}) + \frac{\delta(1-\pi_A)S(\overline{e^{\mathrm{G}}})}{1-\delta} + \frac{1}{1-\delta}\left(c(e^{\mathrm{B}}(\underline{\theta}),\underline{\theta}) - \overline{u}_P\right)\right) = 0
\end{aligned} \tag{84}$$

$$\Rightarrow \quad \eta^{\mathrm{B}}(\underline{\theta}) - \frac{1}{1-\delta}c_e(e^{\mathrm{B}}(\underline{\theta}),\underline{\theta})(\mu^{\mathrm{B}} + \xi - \zeta) = 0 \tag{85}$$

(c)

$$\begin{aligned}
&\lambda_1(\underline{\theta}) + \mu_1^{\mathrm{B}}\frac{\partial K^{\mathrm{B}}(\underline{\theta})}{\partial K^{\mathrm{B}}(\underline{\theta})} + \mu_2^{\mathrm{B}}\frac{\partial L(\underline{\theta})}{\partial K^{\mathrm{B}}(\underline{\theta})} + \mu_3^{\mathrm{B}}\frac{\partial M(\underline{\theta})}{\partial K^{\mathrm{B}}(\underline{\theta})} + \mu_4\frac{\partial(\underline{\theta}-\theta_L)}{\partial K^{\mathrm{B}}(\underline{\theta})} + \mu_5\frac{\partial(\overline{\theta}-\theta_H)}{\partial K^{\mathrm{B}}(\underline{\theta})} \\
&\quad + \mu^{\mathrm{B}}\frac{\partial}{\partial K^{\mathrm{B}}(\underline{\theta})}\left(K^{\mathrm{B}}(\overline{\theta}) + \frac{\delta}{1-\delta}\left(\pi_A S(\overline{e^{\mathrm{G}}}) - \overline{u}_A\right) - \frac{1}{1-\delta}c(e^{\mathrm{B}}(\underline{\theta}),\underline{\theta})\right) \\
&\quad + \xi\frac{\partial}{\partial K^{\mathrm{B}}(\underline{\theta})}\left(L(\overline{\theta}) + \frac{\delta}{1-\delta}\pi_A S(\overline{e^{\mathrm{G}}}) - \frac{1}{1-\delta}\left(c(e^{\mathrm{B}}(\underline{\theta}),\underline{\theta}) + \overline{u}_A\right)\right) \\
&\quad + \zeta\frac{\partial}{\partial K^{\mathrm{B}}(\underline{\theta})}\left(-M(\overline{\theta}) + \frac{\delta(1-\pi_A)S(\overline{e^{\mathrm{G}}})}{1-\delta} + \frac{1}{1-\delta}c(e^{\mathrm{B}}(\underline{\theta}),\underline{\theta}) - \frac{1}{1-\delta}\overline{u}_P\right) = 0
\end{aligned} \tag{86}$$

$$\Rightarrow \quad \lambda_1(\underline{\theta}) + \mu_1^{\mathrm{B}} = 0 \tag{87}$$

(d) Similarly, $\lambda_2(\underline{\theta}) + \mu_2^{\mathrm{B}} = 0$, and

(e) $\lambda_3(\underline{\theta}) + \mu_3^{\mathrm{B}} = 0$

(f)

$$\begin{aligned}
&\mathcal{H}^{\mathrm{B}}(\overline{\theta}) + \mu_1^{\mathrm{B}}\frac{\partial K^{\mathrm{B}}(\underline{\theta})}{\partial\overline{\theta}} + \mu_2^{\mathrm{B}}\frac{\partial L(\underline{\theta})}{\partial\overline{\theta}} + \mu_3^{\mathrm{B}}\frac{\partial M(\underline{\theta})}{\partial\overline{\theta}} + \mu_4\frac{\partial(\underline{\theta}-\theta_L)}{\partial\overline{\theta}} + \mu_5\frac{\partial(\overline{\theta}-\theta_H)}{\partial\overline{\theta}} \\
&+ \mu^{\mathrm{B}}\frac{\partial}{\partial\overline{\theta}}\left(K^{\mathrm{B}}(\overline{\theta}) + \frac{\delta}{1-\delta}\left(\pi_A S(\overline{e^{\mathrm{G}}}) - \overline{u}_A\right) - \frac{1}{1-\delta}c(e^{\mathrm{B}}(\underline{\theta}),\underline{\theta})\right) \\
&+ \xi\frac{\partial}{\partial\overline{\theta}}\left(L(\overline{\theta}) + \frac{\delta}{1-\delta}\pi_A S(\overline{e^{\mathrm{G}}}) - \frac{1}{1-\delta}\left(c(e^{\mathrm{B}}(\underline{\theta}),\underline{\theta}) + \overline{u}_A\right)\right) \\
&+ \zeta\frac{\partial}{\partial\overline{\theta}}\left(-M(\overline{\theta}) + \frac{\delta(1-\pi_A)S(\overline{e^{\mathrm{G}}})}{1-\delta} + \frac{1}{1-\delta}\left(c(e^{\mathrm{B}}(\underline{\theta}),\underline{\theta}) - \overline{u}_P\right)\right) = 0
\end{aligned} \tag{88}$$

$$\Rightarrow \quad \mathcal{H}^{\mathrm{B}}(\overline{\theta}) + \mu_5 = 0 \tag{89}$$

(g)

$$\begin{aligned}
&\eta^{\mathrm{B}}(\overline{\theta}) - \mu_1^{\mathrm{B}}\frac{\partial K^{\mathrm{B}}(\underline{\theta})}{\partial e(\overline{\theta})} - \mu_2^{\mathrm{B}}\frac{\partial L(\underline{\theta})}{\partial e(\overline{\theta})} - \mu_3^{\mathrm{B}}\frac{\partial M(\underline{\theta})}{\partial e(\overline{\theta})} - \mu_4\frac{\partial(\underline{\theta}-\theta_L)}{\partial e(\overline{\theta})} - \mu_5\frac{\partial(\overline{\theta}-\theta_H)}{\partial e(\overline{\theta})} \\
&- \mu^{\mathrm{B}}\frac{\partial}{\partial e(\overline{\theta})}\left(K^{\mathrm{B}}(\overline{\theta}) + \frac{\delta}{1-\delta}\left(\pi_A S(\overline{e^{\mathrm{G}}}) - \overline{u}_A\right) - \frac{1}{1-\delta}c(e^{\mathrm{B}}(\underline{\theta}),\underline{\theta})\right) \\
&- \xi\frac{\partial}{\partial e(\overline{\theta})}\left(L(\overline{\theta}) + \frac{\delta}{1-\delta}\pi_A S(\overline{e^{\mathrm{G}}}) - \frac{1}{1-\delta}\left(c(e^{\mathrm{B}}(\underline{\theta}),\underline{\theta}) + \overline{u}_A\right)\right) \\
&- \zeta\frac{\partial}{\partial e(\overline{\theta})}\left(-M(\overline{\theta}) + \frac{\delta(1-\pi_A)S(\overline{e^{\mathrm{G}}})}{1-\delta} + \frac{1}{1-\delta}c(e^{\mathrm{B}}(\underline{\theta}),\underline{\theta}) - \frac{1}{1-\delta}\overline{u}_P\right) = 0
\end{aligned} \tag{90}$$

$$\Rightarrow \quad \eta^{\mathrm{B}}(\overline{\theta}) = 0 \tag{91}$$

(h)

$$
\begin{aligned}
&\lambda_1(\overline{\theta}) - \mu_1^{\mathrm{B}} \frac{\partial K^{\mathrm{B}}(\underline{\theta})}{\partial K^{\mathrm{B}}(\overline{\theta})} - \mu_2^{\mathrm{B}} \frac{\partial L(\underline{\theta})}{\partial K^{\mathrm{B}}(\overline{\theta})} - \mu_3^{\mathrm{B}} \frac{\partial M(\underline{\theta})}{\partial K^{\mathrm{B}}(\overline{\theta})} - \mu_4 \frac{\partial(\underline{\theta} - \theta_L)}{\partial K^{\mathrm{B}}(\overline{\theta})} - \mu_5 \frac{\partial(\overline{\theta} - \theta_H)}{\partial K^{\mathrm{B}}(\overline{\theta})} \\
&- \mu^{\mathrm{B}} \frac{\partial}{\partial K^{\mathrm{B}}(\overline{\theta})} \left(K^{\mathrm{B}}(\overline{\theta}) + \frac{\delta}{1-\delta} \left(\pi_A S(\overline{e^{\mathrm{G}}}) - \overline{u}_A \right) - \frac{1}{1-\delta} c(e^{\mathrm{B}}(\underline{\theta}), \underline{\theta}) \right) \\
&- \xi \frac{\partial}{\partial K^{\mathrm{B}}(\overline{\theta})} \left(L(\overline{\theta}) + \frac{\delta}{1-\delta} \pi_A S(\overline{e^{\mathrm{G}}}) - \frac{1}{1-\delta} \left(c(e^{\mathrm{B}}(\underline{\theta}), \underline{\theta}) + \overline{u}_A \right) \right) \\
&- \zeta \frac{\partial}{\partial K^{\mathrm{B}}(\overline{\theta})} \left(-M(\overline{\theta}) + \frac{\delta(1-\pi_A) S(\overline{e^{\mathrm{G}}})}{1-\delta} + \frac{1}{1-\delta} c(e^{\mathrm{B}}(\underline{\theta}), \underline{\theta}) - \frac{1}{1-\delta} \overline{u}_P \right) = 0
\end{aligned} \tag{92}
$$

$$
\Rightarrow \quad \lambda_1(\overline{\theta}) - \mu^{\mathrm{B}} = 0 \tag{93}
$$

(i) Similarly,

$$
\lambda_2(\overline{\theta}) - \xi = 0 \tag{94}
$$

(j) , and

$$
\lambda_3(\overline{\theta}) + \zeta = 0. \tag{95}
$$

6. The multipliers must have the following properties:

(a) For the equality boundary constraints, μ_1^{B}, μ_2^{B}, μ_3^{B}, μ_4, μ_5 are constants, and

$$
K^{\mathrm{B}}(\underline{\theta}) = 0, L(\underline{\theta}) = 0, M(\underline{\theta}) = 0, \underline{\theta} = \theta_L, \text{ and } \overline{\theta} = \theta_H \tag{96}
$$

(b) For the inequality boundary constraints, μ^{B}, ξ, ζ are constants, and

$$\mu^{\mathrm{B}} \geq 0 \tag{97}$$

$$K^{\mathrm{B}}(\overline{\theta}) + \frac{\delta}{1-\delta}\left(\pi_A S(\overline{e^{\mathrm{G}}}) - \overline{u}_A\right) - \frac{1}{1-\delta} c(e^{\mathrm{B}}(\underline{\theta}), \underline{\theta}) \geq 0 \tag{98}$$

$$\mu^{\mathrm{B}}\left(K^{\mathrm{B}}(\overline{\theta}) + \frac{\delta}{1-\delta}\left(\pi_A S(\overline{e^{\mathrm{G}}}) - \overline{u}_A\right) - \frac{1}{1-\delta} c(e^{\mathrm{B}}(\underline{\theta}), \underline{\theta})\right) = 0 \tag{99}$$

$$\xi \geq 0 \tag{100}$$

$$L(\overline{\theta}) + \frac{\delta}{1-\delta}\pi_A S(\overline{e^{\mathrm{G}}}) - \frac{1}{1-\delta}\left(c(e^{\mathrm{B}}(\underline{\theta}), \underline{\theta}) + \overline{u}_A\right) \geq 0 \tag{101}$$

$$\xi\left(L(\overline{\theta}) + \frac{\delta}{1-\delta}\pi_A S(\overline{e^{\mathrm{G}}}) - \frac{1}{1-\delta}\left(c(e^{\mathrm{B}}(\underline{\theta}), \underline{\theta}) + \overline{u}_A\right)\right) = 0 \tag{102}$$

$$\zeta \geq 0 \tag{103}$$

$$-M(\overline{\theta}) + \frac{\delta(1-\pi_A)S(\overline{e^{\mathrm{G}}})}{1-\delta} + \frac{1}{1-\delta}\left(c(e^{\mathrm{B}}(\underline{\theta}), \underline{\theta}) - \overline{u}_P\right) \geq 0 \tag{104}$$

$$\zeta\left(-M(\overline{\theta}) + \frac{\delta(1-\pi_A)S(\overline{e^{\mathrm{G}}})}{1-\delta} + \frac{1}{1-\delta}\left(c(e^{\mathrm{B}}(\underline{\theta}), \underline{\theta}) - \overline{u}_P\right)\right) = 0 \tag{105}$$

4.3.3 Observations

1.

$$\dot{\lambda_1}(\theta) = 0\ \forall\theta \text{ from Eq. (75), and } \lambda_1(\overline{\theta}) = \mu^{\mathrm{B}} \text{ from Eq. (93)} \tag{106}$$

$$\dot{\lambda_2}(\theta) = 0\ \forall\theta \text{ from Eq. (76), and } \lambda_2(\overline{\theta}) = \xi \text{ from Eq. (94)} \tag{107}$$

$$\dot{\lambda_3}(\theta) = 0\ \forall\theta \text{ from Eq. (77), and } \lambda_3(\overline{\theta}) = -\zeta \text{ from Eq. (95)} \tag{108}$$

$$\text{Hence, } \lambda_1(\theta) = \mu^{\mathrm{B}}\ \forall\theta,\ \lambda_2(\theta) = \xi\ \forall\theta,\ \lambda_3(\theta) = -\zeta\ \forall\theta \tag{109}$$

2.

$$\eta^{\mathrm{B}}(\theta) = \nu^{\mathrm{B}}(\theta)\ \forall\theta \text{ from Eq. (72) } \Rightarrow \dot{\eta}^{\mathrm{B}}(\theta) = \dot{\nu}^{\mathrm{B}}(\theta)\ \forall\theta \tag{110}$$

3. Substitute for $\eta^{\mathrm{B}}(\theta)$, and $\lambda_1(\theta)$, $\lambda_2(\theta)$, $\lambda_3(\theta)$ in the remaining conditions as follows:

(a) From Eq. (73):

$$\nu^{\mathrm{B}}(\theta) \geq 0,\ -\gamma^{\mathrm{B}}(\theta) \geq 0, \text{ and } \nu^{\mathrm{B}}(\theta)\gamma^{\mathrm{B}}(\theta) = 0 \tag{111}$$

(b) From Eq. (74):

$$\dot{\nu}^{\mathrm{B}}(\theta) = -\left(\pi_A + \frac{\delta\pi_A}{1-\delta}\left(\mu^{\mathrm{B}}+\xi\right) - \zeta\left(\frac{\delta\pi_A}{1-\delta}+1\right)\right)\left(b_e(e^{\mathrm{B}}(\theta)) + c_e(e^{\mathrm{B}}(\theta),\theta)\right)\phi(\theta) + \left(\left(1+\mu^{\mathrm{B}}\frac{\delta}{1-\delta}+\frac{1}{1-\delta}(\xi-\zeta)\right)(1-\Phi(\theta))+\mu^{\mathrm{B}}\right)c_{e\theta}(e^{\mathrm{B}}(\theta),\theta) \tag{112}$$

$$\Rightarrow \left(b_e(e^{\mathrm{B}}(\theta)) + c_e(e^{\mathrm{B}}(\theta),\theta)\right)\phi(\theta) = \frac{\left(\left(1+\frac{1}{1-\delta}(\mu^{\mathrm{B}}+\xi-\zeta)\right)(1-\Phi(\theta))+\mu^{\mathrm{B}}\Phi(\theta)\right)c_{e\theta}(e^{\mathrm{B}}(\theta),\theta) - \dot{\nu}^{\mathrm{B}}(\theta)}{\pi_A + \frac{\delta\pi_A}{1-\delta}\left(\mu^{\mathrm{B}}+\xi-\zeta\right)-\zeta} \tag{113}$$

(c) From Eq. (78):

$$\dot{e^{\mathrm{B}}}(\theta) = \gamma^{\mathrm{B}}(\theta) \tag{114}$$

(d) From Eq. (79):

$$\dot{K}^{\mathrm{B}}(\theta) = \frac{\delta}{1-\delta}\left(\pi_A\left(b(e^{\mathrm{B}}(\theta)) + c(e^{\mathrm{B}}(\theta),\theta)\right)\phi(\theta) - c_\theta(e^{\mathrm{B}}(\theta),\theta)(1-\Phi(\theta))\right) - c_\theta(e^{\mathrm{B}}(\theta),\theta) \tag{115}$$

(e) From Eq. (80):

$$\dot{L}(\theta) = \frac{1}{1-\delta}\left(\delta\pi_A\left(b(e^{\mathrm{B}}(\theta)) + c(e^{\mathrm{B}}(\theta),\theta)\right)\phi(\theta) - c_\theta(e^{\mathrm{B}}(\theta),\theta)(1-\Phi(\theta))\right) \tag{116}$$

(f) From Eq. (81):

$$\dot{M}(\theta) = \left(\frac{\delta\pi_A}{1-\delta}+1\right)\left(b(e^{\mathrm{B}}(\theta),\theta) + c(e^{\mathrm{B}}(\theta),\theta)\right)\phi(\theta) - \frac{1}{1-\delta}c_\theta(e^{\mathrm{B}}(\theta),\theta)(1-\Phi(\theta)) \tag{117}$$

(g) From Eq. (83):

$$\mathcal{H}^{\mathrm{B}}(\underline{\theta}) - \mu_4 + \frac{1}{1-\delta}(\mu^{\mathrm{B}}+\xi-\zeta)c_\theta(e^{\mathrm{B}}(\underline{\theta}),\underline{\theta}) = 0 \tag{118}$$

(h) From Eq. (87):

$$\mu^{\mathrm{B}} + \mu_1^{\mathrm{B}} = 0 \tag{119}$$

and similarly, $\xi + \mu_2^{\mathrm{B}} = 0$, and $-\zeta + \mu_3^{\mathrm{B}} = 0$.

(i) From Eq. (85):

$$\nu^{\mathrm{B}}(\underline{\theta}) = \frac{1}{1-\delta}(\mu^{\mathrm{B}} + \xi - \zeta)c_e(e^{\mathrm{B}}(\underline{\theta}), \underline{\theta}) \tag{120}$$

(j) From Eq. (89):

$$\mathcal{H}^{\mathrm{B}}(\overline{\theta}) + \mu_5 = 0 \tag{121}$$

(k) From Eq. (91):

$$\nu^{\mathrm{B}}(\overline{\theta}) = 0 \tag{122}$$

(l) The properties of multipliers are the same.

4.3.4 Analysis of the bad effort schedule

Observe that μ_1^{B}, μ_2^{B}, μ_3^{B}, μ_4, and μ_5 each appears in only one equation, so the rest of the system can be solved without them. These multipliers, corresponding to the boundary constraints of $K^{\mathrm{B}}(\underline{\theta})$, $L(\underline{\theta})$, $M(\underline{\theta})$, $\underline{\theta}$, and $\overline{\theta}$, respectively, are uninteresting, so we do not solve for them. Hence, the set of equations we will use to solve for the bad effort schedule will be the following:

$$\nu^{\mathrm{B}}(\theta) \geq 0 \tag{123}$$

$$\dot{e}^{\mathrm{B}}(\theta) \leq 0 \tag{124}$$

$$\nu^{\mathrm{B}}(\theta)\dot{e}^{\mathrm{B}}(\theta) = 0 \tag{125}$$

$$\left(b_e(e^{\mathrm{B}}(\theta)) + c_e(e^{\mathrm{B}}(\theta),\theta)\right)\phi(\theta) = \frac{\left(\left(1+\frac{1}{1-\delta}(\mu^{\mathrm{B}}+\xi-\zeta)\right)(1-\Phi(\theta)) + \mu^{\mathrm{B}}\Phi(\theta)\right)c_{e\theta}(e^{\mathrm{B}}(\theta),\theta) - \dot{\nu}^{\mathrm{B}}(\theta)}{\pi_A + \frac{\delta\pi_A}{1-\delta}(\mu^{\mathrm{B}}+\xi-\zeta) - \zeta} \tag{126}$$

$$\mu^{\mathrm{B}} \geq 0 \tag{127}$$

$$K^{\mathrm{B}}(\overline{\theta}) + \frac{\delta}{1-\delta}\left(\pi_A S(\overline{e^{\mathrm{G}}}) - \overline{u}_A\right) - \frac{1}{1-\delta}c(e^{\mathrm{B}}(\underline{\theta}),\underline{\theta}) \geq 0 \tag{128}$$

$$\mu^{\mathrm{B}}\left(K^{\mathrm{B}}(\overline{\theta}) + \frac{\delta}{1-\delta}\left(\pi_A S(\overline{e^{\mathrm{G}}}) - \overline{u}_A\right) - \frac{1}{1-\delta}c(e^{\mathrm{B}}(\underline{\theta}),\underline{\theta})\right) = 0 \tag{129}$$

$$\xi \geq 0 \tag{130}$$

$$L(\overline{\theta}) + \frac{\delta}{1-\delta}\pi_A S(\overline{e^{\mathrm{G}}}) - \frac{1}{1-\delta}\left(c(e^{\mathrm{B}}(\underline{\theta}),\underline{\theta}) + \overline{u}_A\right) \geq 0 \tag{131}$$

$$\xi\left(L(\overline{\theta}) + \frac{\delta}{1-\delta}\pi_A S(\overline{e^{\mathrm{G}}}) - \frac{1}{1-\delta}\left(c(e^{\mathrm{B}}(\underline{\theta}),\underline{\theta}) + \overline{u}_A\right)\right) = 0 \tag{132}$$

$$\zeta \geq 0 \tag{133}$$

$$-M(\overline{\theta}) + \frac{\delta(1-\pi_A)S(\overline{e^{\mathrm{G}}})}{1-\delta} + \frac{1}{1-\delta}\left(c(e^{\mathrm{B}}(\underline{\theta}),\underline{\theta}) - \overline{u}_P\right) \geq 0 \tag{134}$$

$$\zeta\left(-M(\overline{\theta}) + \frac{\delta(1-\pi_A)S(\overline{e^{\mathrm{G}}})}{1-\delta} + \frac{1}{1-\delta}\left(c(e^{\mathrm{B}}(\underline{\theta}),\underline{\theta}) - \overline{u}_P\right)\right) = 0 \tag{135}$$

$$\nu^{\mathrm{B}}(\underline{\theta}) = \frac{1}{1-\delta}(\mu^{\mathrm{B}}+\xi-\zeta)c_e(e^{\mathrm{B}}(\underline{\theta}),\underline{\theta}) \tag{136}$$

$$\nu^{\mathrm{B}}(\overline{\theta}) = 0 \tag{137}$$

Lemma 1. *$\mu^{\mathrm{B}}+\xi-\zeta \geq 0$, $\pi_A + \frac{\delta\pi_A}{1-\delta}\left(\mu^{\mathrm{B}}+\xi-\zeta\right) - \zeta > 0$, and $1+\frac{1}{1-\delta}(\mu^{\mathrm{B}}+\xi-\zeta) - \mu^{\mathrm{B}} > 0$.*

Proof. First, since $\nu^{\mathrm{B}}(\theta) \geq 0$ for all θ, Eq. (136) and Assumption 1 imply that $\mu^{\mathrm{B}}+\xi-\zeta \geq 0$.

Since the lefthand side of Eq. (126) is finite for all θ, the righthand side denominator cannot be zero. Next we show by contradiction that the righthand side denominator is not negative. Assumption 1 implies that the first term in the numerator of the righthand side of Eq. (126) is non-negative. Since the lefthand side of Eq. (126) is strictly positive for all θ, for the righthand side denominator to be strictly negative, it must be that $\dot{\nu}^{\mathrm{B}}(\theta) > 0$ for all θ. But by Eq. (137),

$\int_{\underline{\theta}}^{\overline{\theta}} \dot{\nu}^{\mathrm{B}}(\theta)\, d\theta = 0 - \nu^{\mathrm{B}}(\underline{\theta}) \leq 0$, a contradiction.

Finally, observe that $1 - \pi_A + \frac{1-\delta\pi_A}{1-\delta}(\mu^{\mathrm{B}} + \xi - \zeta) - (\mu^{\mathrm{B}} + \xi - \zeta) + \xi \geq 0$; adding this to $\pi_A + \frac{\delta\pi_A}{1-\delta}\left(\mu^{\mathrm{B}} + \xi - \zeta\right) - \zeta \geq 0$, already established, yields the third fact. □

Next we show that the optimal bad effort schedule is either fully pooling or fully separating.

Lemma 2. *At the solution, either $e^{\mathrm{B}}(\theta) = \hat{e}$ for all θ or $\dot{e}^{\mathrm{B}}(\theta) < 0$ for all θ.*

Proof. First, solve Eq. (126) for $\dot{\nu}^{\mathrm{B}}$:

$$\begin{aligned}\dot{\nu}^{\mathrm{B}}(\theta) = &\left(\left(1 + \frac{1}{1-\delta}(\mu^{\mathrm{B}} + \xi - \zeta)\right)(1 - \Phi(\theta)) + \mu^{\mathrm{B}}\Phi(\theta)\right) c_{e\theta}(e^{\mathrm{B}}(\theta), \theta) \\ &- \left(\pi_A + \frac{\delta\pi_A}{1-\delta}\left(\mu^{\mathrm{B}} + \xi - \zeta\right) - \zeta\right)\left(b_e(e^{\mathrm{B}}(\theta)) + c_e(e^{\mathrm{B}}(\theta), \theta)\right)\phi(\theta)\end{aligned} \tag{138}$$

Now consider any pooling region, in which $e^{\mathrm{B}}(\theta) = \hat{e}$; therein

$$\begin{aligned}\ddot{\nu}^{\mathrm{B}}(\theta) = &\left(\left(1 + \frac{1}{1-\delta}(\mu^{\mathrm{B}} + \xi - \zeta)\right)(1 - \Phi(\theta)) + \mu^{\mathrm{B}}\Phi(\theta)\right) c_{e\theta\theta}(\hat{e}, \theta) \\ &+ \left(-1 - \frac{1}{1-\delta}(\mu^{\mathrm{B}} + \xi - \zeta) + \mu^{\mathrm{B}}\right)\phi(\theta) c_{e\theta}(\hat{e}, \theta) \\ &- \left(\pi_A + \frac{\delta\pi_A}{1-\delta}\left(\mu^{\mathrm{B}} + \xi - \zeta\right) - \zeta\right)\frac{\partial}{\partial\theta}\left(b_e(\hat{e}) + c_e(\hat{e}, \theta)\right)\phi(\theta).\end{aligned} \tag{139}$$

By Assumption 3, the first line on the righthand side is zero. By Lemma 1, Assumption 1, and Assumption 2 the second and third lines are strictly negative. Thus we have shown that $\dot{\nu}^{\mathrm{B}}(\theta)$ is strictly decreasing on any pooling region (where $e^{\mathrm{B}}(\theta)$ is constant). Moreover ν^{B} is continuous, since $\nu^{\mathrm{B}} = \eta^{\mathrm{B}}$ by Equation 110 and η^{B} (a co-state variable) is continuous. Since $\nu^{\mathrm{B}}(\theta) = 0$ implies $\dot{\nu}^{\mathrm{B}}(\theta) = 0$ on separating regions, it follows that $\dot{\nu}^{\mathrm{B}}$ is weakly decreasing everywhere, so ν^{B} is concave.

However, if there are both pooling and separating regions, then ν^{B} must be strictly convex in a neighborhood of any boundary $\hat{\theta}$ between a pooling region and a separating region, because $\nu^{\mathrm{B}}(\hat{\theta} - 2\epsilon) = \nu^{\mathrm{B}}(\hat{\theta} - \epsilon) = 0 < \nu^{\mathrm{B}}(\hat{\theta} + \epsilon)$ for any $\epsilon > 0$ sufficiently small. Since this contradicts concavity, there cannot be both pooling and separating regions. □

Discarding full separation Now we explain why we conjecture that full separation is (almost) never optimal: in this case $\nu^{\mathrm{B}}(\theta) = \dot{\nu}^{\mathrm{B}}(\theta) = 0$ for all θ. Since $c_e(e^{\mathrm{B}}(\underline{\theta}), \underline{\theta}) > 0$ while $\nu^{\mathrm{B}}(\underline{\theta}) = 0$

for full separation, Eq. (136) implies that in this case $\mu^{\mathrm{B}} + \xi - \zeta = 0$. Now Eq. (126) simplifies to:

$$\frac{\left(b_e(e^{\mathrm{B}}(\theta)) + c_e(e^{\mathrm{B}}(\theta), \theta)\right)\phi(\theta)}{c_{e\theta}(e^{\mathrm{B}}(\theta), \theta)} = \frac{1 - \Phi(\theta) + \mu^{\mathrm{B}}\Phi(\theta)}{\pi_A - \zeta} \tag{140}$$

Since the lefthand side is strictly positive by Assumption 1 and Assumption 2 and the righthand side denominator is strictly positive by Lemma 1, evaluating at $\theta = \overline{\theta}$ demonstrates that $\mu^{\mathrm{B}} > 0$—i.e., ICDEB must bind. Since $\mu^{\mathrm{B}} > 0$ and $\xi \geq 0$, it follows that $\zeta \geq \mu^{\mathrm{B}} > 0$, so IRPH also binds. In addition, the denominator on the righthand side must be positive, so $\zeta < \pi_A$. Since this implies $\mu^{\mathrm{B}} < 1$, the righthand side is strictly decreasing in θ. Given μ^{B} and ζ, let $e^{\mathrm{RB}}(\,\cdot\,; \mu^{\mathrm{B}}, \zeta)$ be the unique solution when this equation applies for all θ.

Consider two cases.

1. First, if $\xi = 0$ (IRAH does not bind), then evaluating binding ICDEB and binding IRPH at $e^{\mathrm{RB}}(\,\cdot\,; \mu^{\mathrm{B}}, \zeta)$ yields two equations in μ^{B} and ζ, but we also have $\mu^{\mathrm{B}} = \zeta$. This should lead to a generic contradiction.

2. Next, if $\xi > 0$ (IRAH binds), then evaluating binding ICDEB, binding IRAH, and binding IRPH at $e^{\mathrm{RB}}(\,\cdot\,; \mu^{\mathrm{B}}, \zeta)$ yields three equations in μ^{B}, ξ, and ζ, but we also have $\mu^{\mathrm{B}} + \xi = \zeta$. This should lead to a generic contradiction.

Moreover, the solution to Eq. (140) may not be a decreasing function, also contradicting full separation. This is the case with simple linear-quadratic-uniform functional forms, where Eq. (140) simplifies to

$$\frac{1 + 2\theta e^{\mathrm{B}}(\theta)}{2e^{\mathrm{B}}(\theta)} = \frac{1 - \Phi(\theta) + \mu^{\mathrm{B}}\Phi(\theta)}{\pi_A - \zeta}, \tag{141}$$

to which the solution $e^{\mathrm{RB}}(\,\cdot\,; \mu^{\mathrm{B}}, \zeta)$ is strictly increasing:

$$e^{\mathrm{RB}}(\theta; \mu^{\mathrm{B}}, \zeta) = \frac{\frac{1}{2}(\pi_A - \zeta)}{-(1 + \pi_A - \mu^{\mathrm{B}} - \zeta)\theta + 2 - \mu^{\mathrm{B}}}, \tag{142}$$

$$e^{\mathrm{RB}}_{\theta}(\theta; \mu^{\mathrm{B}}, \zeta) = \frac{\frac{1}{2}(\pi_A - \zeta)(1 + \pi_A - \mu^{\mathrm{B}} - \zeta)}{(1 + \pi_A - \mu^{\mathrm{B}} - \zeta)\theta + 2 - \mu^{\mathrm{B}}} > 0, \tag{143}$$

where the last inequality is implied by $0 < \mu^{\mathrm{B}} \leq \zeta < \pi_A \leq 1$ and $1 \leq \theta \leq 2$.

Full pooling Finally we consider the case of full pooling: in this case $e^{\mathrm{B}}(\theta) = \hat{e}$ for all θ. Since the "first worst" is fully pooling (it cannot be fully separating, as discussed above), if it satisfies the three constraints—ICDEB, IRAH, IRPH—then it is the solution. To solve for the first worst, observe that when all $\mu^{\mathrm{B}} = \xi = \zeta = 0$, Eq. (136) implies that $\nu^{\mathrm{B}}(\underline{\theta}) = 0$. So plug $\mu^{\mathrm{B}} = \xi = \zeta = 0$ into Eq. (138), let $e^{\mathrm{B}}(\theta) = \hat{e}$ for all θ; then integrating yields $\int_{\underline{\theta}}^{\overline{\theta}} \dot{\nu}^{\mathrm{B}}(\theta)\, d\theta = 0$, which fully determines $\hat{e}$. If instead any one of the three constraints binds, then it fully determines $\hat{e}$, so for a given $\overline{e^{\mathrm{G}}}$ generically at most one of these constraints will bind. A straightforward numerical strategy to solve this case is:

1. First identify the first worst; if it satisfies the three constraints then it is the solution; otherwise continue.

2. Next identify three candidates for $\hat{e}$, one that binds each constraint;

3. Add a fourth candidate, $\hat{e} = 0$, in case non-negativity binds;

4. Next eliminate any candidate that violates a constraint;

5. Finally evaluate the objective function at the remaining candidates to determine the optimum.

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Appendix A Analysis of the good problem

Appendix A.1 Necessary conditions

The Hamiltonian for this problem is

$$\begin{aligned}\mathcal{H}^{\mathrm{G}} &= \pi_A \left(g(e^{\mathrm{G}}(\theta)) - c(e^{\mathrm{G}}(\theta),\theta)\right)\phi(\theta) + \eta^{\mathrm{G}}(\theta)\gamma^{\mathrm{G}}(\theta) \\ &\quad + \lambda(\theta)\left(\frac{\delta}{1-\delta}\pi_A \left(g(e^{\mathrm{G}}(\theta)) - c(e^{\mathrm{G}}(\theta),\theta)\right)\phi(\theta) - c_\theta(e^{\mathrm{G}}(\theta),\theta)\right),\end{aligned} \tag{144}$$

and the Lagrangean is

$$\mathcal{L}^{\mathrm{G}} = \mathcal{H}^{\mathrm{G}} - \nu^{\mathrm{G}}(\theta)\gamma^{\mathrm{G}}(\theta). \tag{145}$$

From Theorem 6.5.1 from Léonard and van Long 1992):

1. There exists a piecewise-continuous multiplier $\nu^{\mathsf{G}}(\theta)$ such that, for all θ,

$$\frac{\partial \mathcal{L}^{\mathsf{G}*}}{\partial \gamma^{\mathsf{G}}(\theta)} = 0 \Rightarrow \mathcal{H}^{\mathsf{G}}_{\gamma} - \nu^{\mathsf{G}}(\theta) = 0 \Rightarrow \eta^{\mathsf{G}}(\theta) = \mathcal{H}^{\mathsf{G}}_{\gamma} = \nu^{\mathsf{G}}(\theta), \tag{146}$$

$$\nu^{\mathsf{G}}(\theta) \geq 0, \ -\gamma^{\mathsf{G}}(\theta) \geq 0, \text{ and } \nu^{\mathsf{G}}(\theta)\gamma^{\mathsf{G}}(\theta) = 0 \tag{147}$$

2. The co-state variables $\eta^{\mathsf{G}}(\theta)$, and $\lambda(\theta)$ are continuous, and have piecewise-continuous derivatives satisfying the following conditions:

$$\begin{aligned} \dot{\eta}^{\mathsf{G}}(\theta) &= -\frac{\partial \mathcal{L}^{\mathsf{G}*}}{\partial e^{\mathsf{G}}(\theta)} = -\mathcal{H}^{\mathsf{G}}_{e} \\ &= -\pi_A \left(1 + \frac{\delta}{1-\delta}\lambda(\theta)\right) \left(g_e(e^{\mathsf{G}}(\theta)) - c_e(e^{\mathsf{G}}(\theta), \theta)\right) \phi(\theta) + \lambda(\theta) c_{e\theta}(e^{\mathsf{G}}(\theta), \theta) \end{aligned} \tag{148}$$

$$\dot{\lambda}(\theta) = -\frac{\partial \mathcal{L}^{\mathsf{G}*}}{\partial K^{\mathsf{G}}(\theta)} = -\mathcal{H}^{\mathsf{G}}_{K} = 0 \tag{149}$$

3. The state transitions satisfy

$$\dot{e}^{\mathsf{G}}(\theta) = \gamma^{\mathsf{G}}(\theta) \tag{150}$$

$$\dot{K}^{\mathsf{G}}(\theta) = \frac{\delta}{1-\delta}\pi_A \left(g(e^{\mathsf{G}}(\theta)) - c(e^{\mathsf{G}}(\theta), \theta)\right) \phi(\theta) - c_\theta(e^{\mathsf{G}}(\theta), \theta) \tag{151}$$

4. The Lagrangean $\mathcal{L}^{\mathsf{G}}(e^{\mathsf{G}}(\theta)^*, K^{\mathsf{G}}(\theta)^*, \eta^{\mathsf{G}}(\theta), \lambda(\theta), \nu^{\mathsf{G}}(\theta), \theta) = \psi^{\mathsf{G}}(\theta)$ is a continuous function of θ. On each interval of continuity of $\gamma^{\mathsf{G}^*}(\theta)$, $\psi^{\mathsf{G}}(\theta)$ is differentiable, and $\psi^{\mathsf{G}'}(\theta) \equiv \frac{d\mathcal{L}^{\mathsf{G}*}}{d\theta} = \frac{\partial \mathcal{L}^{\mathsf{G}*}}{\partial \theta}$.

5. Transversality conditions:

   (a)

$$\begin{aligned} &\mathcal{H}^{\mathsf{G}}(\underline{\theta}) - \mu^{\mathsf{G}}_1 \frac{\partial K^{\mathsf{G}}(\underline{\theta})}{\partial \underline{\theta}} - \mu^{\mathsf{G}}_2 \frac{\partial}{\partial \underline{\theta}}(\underline{\theta} - \theta_L) - \mu^{\mathsf{G}}_3 \frac{\partial}{\partial \underline{\theta}}(\overline{\theta} - \theta_H) \\ &\quad - \mu^{\mathsf{G}} \frac{\partial}{\partial \underline{\theta}} \begin{pmatrix} K^{\mathsf{G}}(\overline{\theta}) - c(e^{\mathsf{G}}(\underline{\theta}), \underline{\theta}) \\ - \frac{\delta}{1-\delta}\left(\pi_A S(\overline{e^{\mathsf{B}}}) + c(\overline{e^{\mathsf{B}}}(\underline{\theta}), \underline{\theta}) + c_\theta(\overline{e^{\mathsf{B}}}(\theta), \theta) + \overline{u}_A\right) \end{pmatrix} = 0 \end{aligned} \tag{152}$$

$$\Rightarrow \quad \mathcal{H}^{\mathsf{G}}(\underline{\theta}) + \mu^{\mathsf{G}} c_\theta(e^{\mathsf{G}}(\underline{\theta}), \underline{\theta}) - \mu^{\mathsf{G}}_2 = 0 \tag{153}$$

(b)

$$\eta^{\mathrm{G}}(\underline{\theta}) + \mu_1^{\mathrm{G}} \frac{\partial K^{\mathrm{G}}(\underline{\theta})}{\partial e(\underline{\theta})} + \mu_2^{\mathrm{G}} \frac{\partial}{\partial e(\underline{\theta})}(\underline{\theta} - \theta_L) + \mu_3^{\mathrm{G}} \frac{\partial}{\partial e(\underline{\theta})}(\overline{\theta} - \theta_H)$$
$$- \mu^{\mathrm{G}} \frac{\partial}{\partial e(\underline{\theta})} \begin{pmatrix} K^{\mathrm{G}}(\overline{\theta}) - c(e^{\mathrm{G}}(\underline{\theta}), \underline{\theta}) \\ - \frac{\delta}{1-\delta} \left(\pi_A S(\overline{e^{\mathrm{B}}}) + c(\overline{e^{\mathrm{B}}}(\underline{\theta}), \underline{\theta}) + c_\theta(\overline{e^{\mathrm{B}}}(\theta), \theta) + \overline{u}_A \right) \end{pmatrix} = 0 \tag{154}$$
$$\Rightarrow \quad \eta^{\mathrm{G}}(\underline{\theta}) - \mu^{\mathrm{G}}\, c_e(e^{\mathrm{G}}(\underline{\theta}), \underline{\theta}) = 0 \tag{155}$$

(c)

$$\lambda(\underline{\theta}) + \mu_1^{\mathrm{G}} \frac{\partial K^{\mathrm{G}}(\underline{\theta})}{\partial K^{\mathrm{G}}(\underline{\theta})} + \mu_2^{\mathrm{G}} \frac{\partial}{\partial K^{\mathrm{G}}(\underline{\theta})}(\underline{\theta} - \theta_L) + \mu_3^{\mathrm{G}} \frac{\partial}{\partial K^{\mathrm{G}}(\underline{\theta})}(\overline{\theta} - \theta_H)$$
$$- \mu^{\mathrm{G}} \frac{\partial}{\partial K^{\mathrm{G}}(\underline{\theta})} \begin{pmatrix} K^{\mathrm{G}}(\overline{\theta}) - c(e^{\mathrm{G}}(\underline{\theta}), \underline{\theta}) \\ - \frac{\delta}{1-\delta} \left(\pi_A S(\overline{e^{\mathrm{B}}}) + c(\overline{e^{\mathrm{B}}}(\underline{\theta}), \underline{\theta}) + c_\theta(\overline{e^{\mathrm{B}}}(\theta), \theta) + \overline{u}_A \right) \end{pmatrix} = 0 \tag{156}$$
$$\Rightarrow \quad \lambda(\underline{\theta}) + \mu_1^{\mathrm{G}} = 0 \tag{157}$$

(d)

$$\mathcal{H}^{\mathrm{G}}(\overline{\theta}) + \mu_1^{\mathrm{G}} \frac{\partial K^{\mathrm{G}}(\underline{\theta})}{\partial \overline{\theta}} + \mu_2^{\mathrm{G}} \frac{\partial}{\partial \overline{\theta}}(\underline{\theta} - \theta_L) + \mu_3^{\mathrm{G}} \frac{\partial}{\partial \overline{\theta}}(\overline{\theta} - \theta_H)$$
$$- \mu^{\mathrm{G}} \frac{\partial}{\partial \overline{\theta}} \begin{pmatrix} K^{\mathrm{G}}(\overline{\theta}) - c(e^{\mathrm{G}}(\underline{\theta}), \underline{\theta}) \\ - \frac{\delta}{1-\delta} \left(\pi_A S(\overline{e^{\mathrm{B}}}) + c(\overline{e^{\mathrm{B}}}(\underline{\theta}), \underline{\theta}) + c_\theta(\overline{e^{\mathrm{B}}}(\theta), \theta) + \overline{u}_A \right) \end{pmatrix} = 0 \tag{158}$$
$$\Rightarrow \quad \mathcal{H}^{\mathrm{G}}(\overline{\theta}) + \mu_3^{\mathrm{G}} = 0 \tag{159}$$

(e)

$$\eta^{\mathrm{G}}(\overline{\theta}) + \mu_1^{\mathrm{G}} \frac{\partial K^{\mathrm{G}}(\underline{\theta})}{\partial e(\overline{\theta})} + \mu_2^{\mathrm{G}} \frac{\partial}{\partial e(\overline{\theta})}(\underline{\theta} - \theta_L) + \mu_3^{\mathrm{G}} \frac{\partial}{\partial e(\overline{\theta})}(\overline{\theta} - \theta_H)$$
$$- \mu^{\mathrm{G}} \frac{\partial}{\partial e(\overline{\theta})} \begin{pmatrix} K^{\mathrm{G}}(\overline{\theta}) - c(e^{\mathrm{G}}(\underline{\theta}), \underline{\theta}) \\ - \frac{\delta}{1-\delta} \left(\pi_A S(\overline{e^{\mathrm{B}}}) + c(\overline{e^{\mathrm{B}}}(\underline{\theta}), \underline{\theta}) + c_\theta(\overline{e^{\mathrm{B}}}(\theta), \theta) + \overline{u}_A \right) \end{pmatrix} = 0 \tag{160}$$
$$\Rightarrow \quad \eta^{\mathrm{G}}(\overline{\theta}) = 0 \tag{161}$$

(f)

$$\lambda(\overline{\theta}) + \mu_1^{\mathrm{G}} \frac{\partial K^{\mathrm{G}}(\underline{\theta})}{\partial K^{\mathrm{G}}(\overline{\theta})} + \mu_2^{\mathrm{G}} \frac{\partial}{\partial K^{\mathrm{G}}(\overline{\theta})} (\underline{\theta} - \theta_L) + \mu_3^{\mathrm{G}} \frac{\partial}{\partial K^{\mathrm{G}}(\overline{\theta})} (\overline{\theta} - \theta_H)$$
$$- \mu^{\mathrm{G}} \frac{\partial}{\partial K^{\mathrm{G}}(\overline{\theta})} \left(\begin{array}{l} K^{\mathrm{G}}(\overline{\theta}) - c(e^{\mathrm{G}}(\underline{\theta}), \underline{\theta}) \\ - \frac{\delta}{1-\delta} \left(\pi_A S(\overline{e^{\mathrm{B}}}) + c(\overline{e^{\mathrm{B}}}(\underline{\theta}), \underline{\theta}) + c_\theta(\overline{e^{\mathrm{B}}}(\theta), \theta) + \overline{u}_A \right) \end{array} \right) = 0 \tag{162}$$

$$\Rightarrow \quad \lambda(\overline{\theta}) - \mu^{\mathrm{G}} = 0 \tag{163}$$

6. Also the multipliers have the following properties:

   (a) For the equality boundary constraints, μ_1^{G}, μ_2^{G}, μ_3^{G} are constants, and

$$K^{\mathrm{G}}(\underline{\theta}) = 0 \text{ , } \underline{\theta} - \theta_L = 0, \text{ and } \overline{\theta} - \theta_H = 0 \tag{164}$$

   (b) For the inequality boundary constraint μ^{G} is constant, and

$$\mu^{\mathrm{G}} \geq 0$$
$$K^{\mathrm{G}}(\overline{\theta}) - \frac{\delta}{1-\delta} \left(\pi_A S(\overline{e^{\mathrm{B}}}) + c(\overline{e^{\mathrm{B}}}(\underline{\theta}), \underline{\theta}) + c_\theta(\overline{e^{\mathrm{B}}}(\theta), \theta) + \overline{u}_A \right) - c(e^{\mathrm{G}}(\underline{\theta}), \underline{\theta}) \geq 0$$
$$\mu^{\mathrm{G}} \left(K^{\mathrm{G}}(\overline{\theta}) - \frac{\delta}{1-\delta} \left(\pi_A S(\overline{e^{\mathrm{B}}}) + c(\overline{e^{\mathrm{B}}}(\underline{\theta}), \underline{\theta}) + c_\theta(\overline{e^{\mathrm{B}}}(\theta), \theta) + \overline{u}_A \right) - c(e^{\mathrm{G}}(\underline{\theta}), \underline{\theta}) \right) = 0 \tag{165}$$

Appendix A.2 Observations

1.

$$\dot{\lambda}(\theta) = 0 \forall \theta \text{ from Eq. (149) and } \lambda(\overline{\theta}) = \mu^{\mathrm{G}} \text{ from Eq. (163)} \tag{166}$$

2.

$$\eta^{\mathrm{G}}(\theta) = \nu^{\mathrm{G}}(\theta) \forall \theta \text{ from Eq. (146)} \Rightarrow \dot{\eta}^{\mathrm{G}}(\theta) = \dot{\nu}^{\mathrm{G}}(\theta) \forall \theta \tag{167}$$

3. Substitute for $\eta^{\mathrm{G}}(\theta)$, and $\lambda(\theta)$ in the remaining conditions as follows:

(a) From Eq. (147):

$$\nu^{\mathrm{G}}(\theta) \geq 0\,, \, -\gamma^{\mathrm{G}}(\theta) \geq 0, \text{and } \nu^{\mathrm{G}}(\theta)\gamma^{\mathrm{G}}(\theta) = 0 \tag{168}$$

(b) From Eq. (148):

$$\begin{aligned}
-\dot{\nu^{\mathrm{G}}}(\theta) = \pi_A \left(1 + \mu^{\mathrm{G}}\frac{\delta}{1-\delta}\right) \left(g_e(e^{\mathrm{G}}(\theta)) - c_e(e^{\mathrm{G}}(\theta),\theta)\right) \phi(\theta) - \mu c_{e\theta}(e^{\mathrm{G}}(\theta),\theta) \Rightarrow \\
\Rightarrow \pi_A \left(1 + \mu^{\mathrm{G}}\frac{\delta}{1-\delta}\right) \left(g_e(e^{\mathrm{G}}(\theta)) - c_e(e^{\mathrm{G}}(\theta),\theta)\right) \phi(\theta) = \mu c_{e\theta}(e^{\mathrm{G}}(\theta),\theta) - \dot{\nu}_G(\theta) \Rightarrow \\
\left(g_e(e^{\mathrm{G}}(\theta)) - c_e(e^{\mathrm{G}}(\theta),\theta)\right) \phi(\theta) = \frac{\mu^{\mathrm{G}}}{\pi_A \left(1 + \mu^{\mathrm{G}}\frac{\delta}{1-\delta}\right)} \left(c_{e\theta}(e^{\mathrm{G}}(\theta),\theta) - \frac{1}{\mu^{\mathrm{G}}}\dot{\nu}^{\mathrm{G}}(\theta)\right)
\end{aligned} \tag{169}$$

(c) From Eq. (150):

$$\dot{e}^{\mathrm{G}}(\theta) = \gamma^{\mathrm{G}}(\theta) \tag{170}$$

(d) From Eq. (151):

$$\dot{K^{\mathrm{G}}}(\theta) = \frac{\delta}{1-\delta}\pi_A \left[g(e^{\mathrm{G}}(\theta)) - c(e^{\mathrm{G}}(\theta),\theta)\right] \phi(\theta) - c_\theta(e^{\mathrm{G}}(\theta),\theta) \tag{171}$$

(e) From Eq. (153):

$$\begin{aligned}
&\mathcal{H}^{\mathrm{G}}(\underline{\theta}) + \mu^{\mathrm{G}} c_\theta(e^{\mathrm{G}}(\underline{\theta}),\underline{\theta}) - \mu_2^{\mathrm{G}} = 0 \Rightarrow \\
&\Rightarrow \left[g(e^{\mathrm{G}}(\underline{\theta})) - c(e^{\mathrm{G}}(\underline{\theta}),\underline{\theta})\right] \phi(\underline{\theta})\pi_A + \nu^{\mathrm{G}}(\underline{\theta})\gamma^{\mathrm{G}}(\underline{\theta}) + \\
&+ \mu^{\mathrm{G}} \left[\frac{\delta}{1-\delta}\pi_A \left[g(e^{\mathrm{G}}(\underline{\theta})) - c(e^{\mathrm{G}}(\underline{\theta}),\underline{\theta})\right] \phi(\underline{\theta}) - c_\theta(e^{\mathrm{G}}(\underline{\theta}),\underline{\theta})\right] + \\
&+ \mu^{\mathrm{G}} c_\theta(e^{\mathrm{G}}(\underline{\theta}),\underline{\theta}) - \mu_2^{\mathrm{G}} = 0 \Rightarrow \\
&\Rightarrow \pi_A \left[g(e^{\mathrm{G}}(\underline{\theta})) - c(e^{\mathrm{G}}(\underline{\theta}),\underline{\theta})\right] \phi(\underline{\theta}) \left(1 + \mu^{\mathrm{G}}\frac{\delta}{1-\delta}\right) + \nu^{\mathrm{G}}(\underline{\theta})\gamma^{\mathrm{G}}(\underline{\theta}) - \mu_2^{\mathrm{G}} = 0
\end{aligned} \tag{172}$$

(f) From Eq. (157):

$$\mu^{\mathrm{G}} + \mu_1^{\mathrm{G}} = 0 \tag{173}$$

(g) From Eq. (155):

$$\nu^{\mathrm{G}}(\underline{\theta}) - \mu^{\mathrm{G}} c_e(e^{\mathrm{G}}(\underline{\theta}), \underline{\theta}) = 0 \Rightarrow \nu^{\mathrm{G}}(\underline{\theta}) = \mu^{\mathrm{G}} c_e(e^{\mathrm{G}}(\underline{\theta}), \underline{\theta}) \tag{174}$$

(h) From Eq. (159):

$$\begin{aligned}
&\mathcal{H}^{\mathrm{G}}(\overline{\theta}) + \mu_3^{\mathrm{G}} = 0 \Rightarrow \\
&\Rightarrow \left[g(e^{\mathrm{G}}(\overline{\theta})) - c(e^{\mathrm{G}}(\overline{\theta}), \overline{\theta})\right] \phi(\overline{\theta})\pi_A + \mu^{\mathrm{G}} \left[\frac{\delta}{1-\delta}\pi_A \left[g(e^{\mathrm{G}}(\overline{\theta})) - c(e^{\mathrm{G}}(\overline{\theta}), \overline{\theta})\right] \phi(\overline{\theta}) - c_\theta(e^{\mathrm{G}}(\overline{\theta}), \overline{\theta})\right] \\
&+ \nu^{\mathrm{G}}(\overline{\theta})\gamma(\overline{\theta}) + \mu_3^{\mathrm{G}} = 0 \Rightarrow \\
&\Rightarrow \left[g(e^{\mathrm{G}}(\overline{\theta})) - c(e^{\mathrm{G}}(\overline{\theta}), \overline{\theta})\right] \phi(\overline{\theta})\pi_A \left(1 + \mu^{\mathrm{G}} \frac{\delta}{1-\delta}\right) + \nu^{\mathrm{G}}(\overline{\theta})\gamma(\overline{\theta}) - \mu^{\mathrm{G}} c_\theta(e^{\mathrm{G}}(\overline{\theta}), \overline{\theta}) + \mu_3^{\mathrm{G}} = 0
\end{aligned} \tag{175}$$

(i) From Eq. (161)

$$\nu^{\mathrm{G}}(\overline{\theta}) = 0 \tag{176}$$

(j) The properties of the multipliers remain the same.

Appendix A.3 Consolidated necessary conditions

Observe that μ_1^{G}, μ_2^{G}, and μ_3^{G} each appears in only one equation, so the rest of the system can be solved without them. These multipliers, corresponding to the boundary constraints of $K^{\mathrm{G}}(\underline{\theta})$, $\underline{\theta}$, and $\overline{\theta}$, respectively, are uninteresting, so we do not solve for them. Hence, the set of equations we will use to solve for the good effort schedule will be the following:

$$\nu^{\mathrm{G}}(\theta) \geq 0, \tag{177}$$
$$\dot{e}^{\mathrm{G}}(\theta) \leq 0, \tag{178}$$
$$\nu^{\mathrm{G}}(\theta)\dot{e}^{\mathrm{G}}(\theta) = 0, \tag{179}$$
$$\left(g_e(e^{\mathrm{G}}(\theta)) - c_e(e^{\mathrm{G}}(\theta),\theta)\right)\phi(\theta) = \frac{\mu^{\mathrm{G}} c_{e\theta}(e^{\mathrm{G}}(\theta),\theta) - \dot{\nu}^{\mathrm{G}}(\theta)}{\pi_A\left(1+\mu^{\mathrm{G}}\frac{\delta}{1-\delta}\right)} \tag{180}$$
$$\mu^{\mathrm{G}} \geq 0, \tag{181}$$
$$K^{\mathrm{G}}(\overline{\theta}) - \frac{\delta}{1-\delta}\left(\pi_A S(\overline{e^{\mathrm{B}}}) + c(\overline{e^{\mathrm{B}}}(\underline{\theta}),\underline{\theta}) + c_\theta(\overline{e^{\mathrm{B}}}(\theta),\theta) + \overline{u}_A\right) - c(e^{\mathrm{G}}(\underline{\theta}),\underline{\theta}) \geq 0, \tag{182}$$
$$\mu^{\mathrm{G}}\left(K^{\mathrm{G}}(\overline{\theta}) - \frac{\delta}{1-\delta}\left(\pi_A S(\overline{e^{\mathrm{B}}}) + c(\overline{e^{\mathrm{B}}}(\underline{\theta}),\underline{\theta}) + c_\theta(\overline{e^{\mathrm{B}}}(\theta),\theta) + \overline{u}_A\right) - c(e^{\mathrm{G}}(\underline{\theta}),\underline{\theta})\right) = 0, \tag{183}$$
$$\nu^{\mathrm{G}}(\underline{\theta}) = \mu^{\mathrm{G}} c_e(e^{\mathrm{G}}(\underline{\theta}),\underline{\theta}), \tag{184}$$
$$\nu^{\mathrm{G}}(\overline{\theta}) = 0. \tag{185}$$

Conclusions

1. $\mu^{\mathrm{G}} = 0$: then the ICDEG is slack at the solution, and thus it suffices to maximize $z_A^H(e^{\mathrm{G}}, \overline{e^{\mathrm{B}}})$ subject to the monotonicity constraint, $-\gamma^{\mathrm{G}}(\theta) \geq 0$. Because first-best effort e^{FB} is decreasing, it both maximizes z_A^H and satisfies monotonicity. Thus, $e^{\mathrm{G}} = e^{FB}$.

2. $\mu^{\mathrm{G}} > 0$: then the ICDEG binds at the solution.

   (a) Define $e^{\mathrm{RG}}(\theta)$ as the unique solution to

   $$\left(g_e(e(\theta)) - c_e(e(\theta),\theta)\right)\phi(\theta) = \frac{\mu^{\mathrm{G}}}{\pi_A\left(1+\mu^{\mathrm{G}}\frac{\delta}{1-\delta}\right)} c_{e\theta}(e(\theta),\theta). \tag{186}$$

   Under our assumptions, $e^{\mathrm{RG}}(\theta)$ is decreasing in θ, and $e^{\mathrm{RG}}(\theta) < e^{FB}(\theta)\forall\theta$.

   (b) On any interval value where the solution $e^{\mathrm{G}}(\theta)$ is strictly decreasing in θ, $e^{\mathrm{G}}(\theta) = e^{\mathrm{RG}}(\theta)$. (NOTE: proof same as Levin's)

   (c) If for some $\hat{\theta}$, $\dot{e}^{\mathrm{G}}(\hat{\theta}) < 0$, then $e^{\mathrm{G}}(\theta)$ is decreasing on $[\hat{\theta},\overline{\theta}]$. (NOTE: proof same as Levin's)

   (d) If ICDEG is binding, $\mu^{\mathrm{G}} > 0$, and from assumptions we also have that $c_e > 0$. Since at the optimum, $\nu^{\mathrm{G}}(\underline{\theta}) = \mu^{\mathrm{G}} c_e(e^{\mathrm{G}}(\underline{\theta}),\underline{\theta})$, it follows that $\nu^{\mathrm{G}}(\underline{\theta}) > 0$. Hence, from complementary slackness ($\nu^{\mathrm{G}}(\theta)\gamma^{\mathrm{G}}(\theta) = 0$), we have that $\gamma^{\mathrm{G}}(\theta) = \dot{e}^{\mathrm{G}}(\theta) = 0$. So, there must be pooling of the most efficient types. Combined with points (b), (c); two possibilities arise:

i. *Partial pooling*: $e^{\mathrm{G}}(\theta)$ will be constant below some $\hat{\theta} \in (\underline{\theta}, \overline{\theta})$, and decreasing above it; i.e.

$$e^{\mathrm{G}}(\theta) = \begin{cases} e^{\mathrm{RG}}(\theta), & \forall \theta \geq \hat{\theta} \\ \hat{e^{\mathrm{G}}}, & \forall \theta \leq \hat{\theta} \end{cases} \tag{187}$$

By continuity we have that $\hat{e^{\mathrm{G}}} = e^{\mathrm{RG}}(\hat{\theta})$. To find the "cut-off" type $\hat{\theta}$, observe that $\nu^{\mathrm{G}}(\underline{\theta}) = \mu^{\mathrm{G}} c_e(e^{\mathrm{G}}(\underline{\theta}), \underline{\theta})$, and $\nu^{\mathrm{G}}(\hat{\theta}) = 0$. If we integrate Eq. (169) from $\underline{\theta}$ to $\hat{\theta}$, and substitute the boundary conditions, we have:

$$\begin{aligned} &\int_{\underline{\theta}}^{\hat{\theta}} \left[g_e(e^{\mathrm{G}}(\theta)) - c_e(e^{\mathrm{G}}(\theta), \theta)\right] \phi(\theta) d\theta = \frac{\mu^{\mathrm{G}}}{\pi_A \left(1 + \mu^{\mathrm{G}} \frac{\delta}{1-\delta}\right)} \int_{\underline{\theta}}^{\hat{\theta}} \left[c_{e\theta}(e^{\mathrm{G}}(\theta), \theta) - \frac{1}{\mu^{\mathrm{G}}} \dot{\nu}^{\mathrm{G}}(\theta)\right] d\theta = \\ &= \frac{\mu^{\mathrm{G}}}{\pi_A \left(1 + \mu^{\mathrm{G}} \frac{\delta}{1-\delta}\right)} \left[c_e(e^{\mathrm{G}}(\hat{\theta}), \hat{\theta}) - \frac{1}{\mu^{\mathrm{G}}} \nu^{\mathrm{G}}(\hat{\theta}) - c_e(e^{\mathrm{G}}(\underline{\theta}), \underline{\theta}) + \frac{1}{\mu^{\mathrm{G}}} \nu^{\mathrm{G}}(\underline{\theta})\right] = \\ &= \frac{\mu^{\mathrm{G}}}{\pi_A \left(1 + \mu^{\mathrm{G}} \frac{\delta}{1-\delta}\right)} c_e(e^{\mathrm{G}}(\hat{\theta}), \hat{\theta}). \end{aligned} \tag{188}$$

From the assumptions in the primitives, there is at most one $\hat{\theta} \in (\underline{\theta}, \overline{\theta})$ that satisfies this condition.

ii. *Full pooling*: $e^{\mathrm{G}}(\theta) = \hat{e}$ for all θ. Simply solve for $\hat{e}$ from the binding ICDEG constraint.